Examples and Variations

I $\displaystyle\int_{[0,\pi]\times[1,2]} \frac{\cos(x)}{y}\, dA(x,y)$

$$= \int_0^{\pi} \cos(x)\,dx \int_1^2 \frac{dy}{y}$$

$$= \cancel{1} \ln 2$$

II $\displaystyle\iint_{[1,2]\times[2,3]} e^x y\, dA(x,y) = \int_1^2 e^x dx \int_2^3 y\,dy$

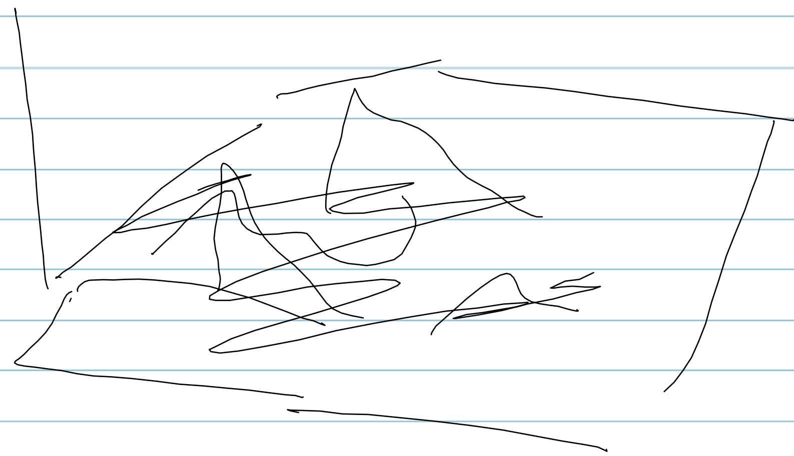
$$= (e^2 - e^1)\left(\frac{9}{2} - \frac{4}{2}\right)$$

In general

$$\int_{[a,b]\times[c,d]} f(x)g(y)\, dA(x,y) = \int_a^b f(x)\,dx \int_c^d g(y)\,dy \quad (*)$$

(v) is important in probability

<u>Pb</u> Find ~~average~~ average height



(Raise ocean by one meter = how much water)

Solution
$$\boxed{\int_D f(x,y)\,\frac{dA(x,y)}{\text{Area}(D)} = \text{Average}(f)}$$

<u>Thm</u> $f(x,y) = F(x)\,G(y) \qquad D = [a,b] \times [c,d]$

$$\boxed{\text{Av}(F \cdot G) = \text{Av}(F)\,\text{Av}(G)}$$

Ex

III $\displaystyle\int_{[1,2]\times[3,4]} \sqrt{x+y}\, dA(x,y)$

IV $\displaystyle\int_{0\le y\le x\le 1} (1+x^2)^{1/3}\, dA(x,y) \quad \left(= \int \mathbb{1}_D f(x,y)\, dA\right)$

cont

$u = 1+x^2 \quad du = 2x\,dx$

$$= \int_0^1 x(1+x^2)^{1/3}\, dx \qquad \text{(Fubini)}$$

$$= \frac{1}{2}\int_1^2 u^{1/3}\, du$$

$$= \frac{1}{2}\, \frac{3}{4} u^{4/3}\Big|_1^2 = \frac{3}{8}\left(2^{4/3}-1\right)$$

$$\int_{0\le y\le x^2\le 1} (2+x^3)^{1/3}\, dA(x,y) = \cdots$$

integrable (why?)

ct

$u = x^3$

$du = 3x^2\,dx$

$$= \int \mathbb{1}_D f(x,y)\, dA$$

$$= \int_0^1 x^2(2+x^3)^{1/3}\, dx = \frac{1}{3}\int_2^3 u^{1/3}\, du = \frac{1}{3}\,\frac{3}{4}\left(3^{4/3}-2^{4/3}\right)$$

$\square$

Converse of Fubini $f: [a,b]\times[c,d] \longrightarrow \mathbb{R}$ positive

and for every x $f^x_{\cdot}(y) = f(x,y)$ is integrable

such that

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \int_c^d f_x(y)\,dy\,dx < \infty$$

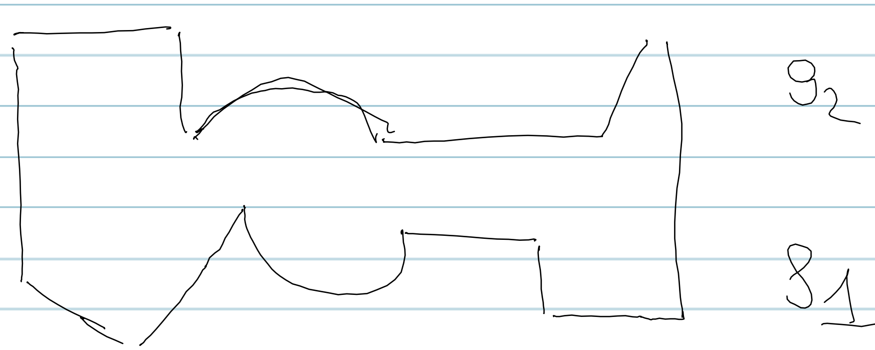
Then $\int f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Helpful in proving that the following domains are "integrable" (1_D

type I $\{(x,y):\ g_1(x) \le y \le g_2(x)$ $\}$
$a \le x \le b$

g_1, g_2 with finitely many discontinuities

Ex



type II

$$D = \{(x,y) : \begin{array}{l} g_1(y) \le x \le g_2(y) \\ -c \le y \le d \end{array}\}$$

In 3D

type I

$$D_{I} = \{(x,y,z) : g_1(x,y) \le z \le g_2(x,y)\}$$

type II

$$D_{II} = \{(x,z) : g_1(x,z) \le y \le g_2(x,z)\}$$

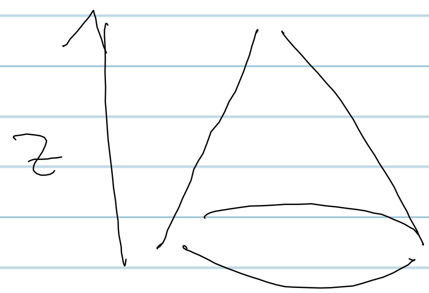
$$D_{III} = \{ \quad g_1(y,z) \le x \le g_2(y,z)\}$$

Converse (used in previous examples)

Remark D type I, II, fct

Then $1_D f$ integrable (and Fubini applies)

Extension of Fubini / Cavalieri



volume of cone

$$D = \{ (x,y,z) : 0 \le z \le 1 \quad \sqrt{x^2+y^2} \le 1-z \}$$

For every fixed z the circle

$$D(z) = \{ (x,y) : x^2+y^2 \le (1-z)^2 \}$$

has an area

$$1_{D(z)}(x,y) \quad \text{is integrable}$$

Converse of Fubini

$$\int_{[0,1]^2 \times [0,1]} 1_C(x,y,z)\, dV(x,y,z) \qquad \text{if finite}$$

$$= \int_0^1 \int_{[-1,1]^2} 1_{D(z)}(x,y)\, dA(x,y)\, dz$$

Of course

$$\int_0^1 \int_{[-1,1]} 1_{D(z)} \, dA \quad dz$$

$$= \int_0^1 \text{area}(D(z)) \quad dz$$

$$= \int_0^1 \pi (1-z)^2 \, dz$$

$$= \pi \int_0^1 z^2 \, dz$$

$$= \frac{\pi}{3}$$

~~Archimedes~~

Euclid's elements and even before

$$\boxed{\text{Volume}(\text{Cone}) = \tfrac{1}{3} \text{ volume of surrounding cylinder}}$$

Important Check assumptions

Pb Find volume between plane

$x = \pm 1$ $y = 0, \pi$, $z = 0$ and

$z = 1 + e^x \sin y$