Conservative vector fields

<u>Def</u> F is called conservative if there exists f with $\nabla f = F$

<u>Thm ($\mathbb{R}^2$)</u> F conservative iff $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

$\quad \shortparallel$

$\begin{pmatrix} P \\ Q \end{pmatrix}$

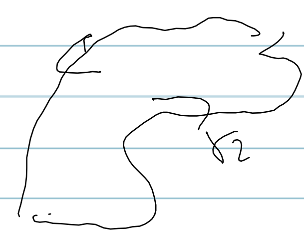
<u>pf</u> $F = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$ then $\frac{\partial^2 f}{\partial x y} = \frac{\partial^2 f}{\partial y x}$ implies

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

On the other hand <u>assume</u> $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ Define.

$R\ F(0) = 0$

$f\begin{pmatrix} x \\ y \end{pmatrix} = \int_0^{\begin{pmatrix} x \\ y \end{pmatrix}} F \cdot T \, ds$



any path

(well defined bc $\int_{\partial D} F \cdot T \, ds = \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = 0$

$\int_{\gamma_2 \cup \{-\gamma_1\}} F \cdot T \, ds = \int_{(0)\gamma_1}^{\begin{pmatrix} x \\ y \end{pmatrix}} F \cdot T \, ds - \int_{\gamma_2} F \cdot T \, ds$

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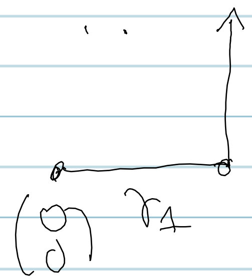
Claim $\frac{\partial f}{\partial x} = P$

Pf



$\gamma_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \quad 0 \le t \le x$

$\gamma_2(t) = \begin{pmatrix} x \\ t \end{pmatrix} \quad 0 \le t \le y$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \int_0^x P(t,0)\,dt + \int_0^y Q(x,t)\,dt$$

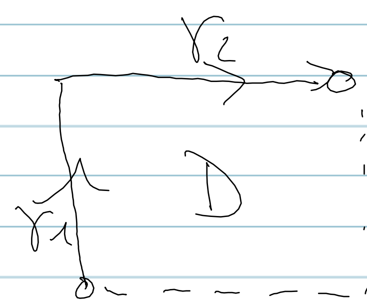
$$\frac{\partial f}{\partial x} = P(x,0) + \int_0^y \frac{\partial Q}{\partial x}(x,t)\,dt$$

$$= P(x,0) + \int_0^y \frac{\partial P}{\partial y}(x,t)\,dt$$

$$= P(x,0) + P(x,y) - P(x,0) = P(x,y)$$

Similar

$\frac{\partial f}{\partial y} = Q(x,y)$ here



(Note same f !)

$$\boxed{\int_D = 0}$$

∎

Thm ($\mathbb{R}^3$) $\vec F$ is conservative iff $\mathrm{curl}(\vec F) = 0$

Proof $\mathrm{curl}\,\nabla f = 0$ easy

For the converse we need

$$\int_{\gamma_1} F\cdot T\,ds = \int_{\gamma_2} F\cdot T\,ds$$

whenever $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(b) = \gamma_2(b)$ have same endpoints

(Remark We use a non-trivial fact:

For every closed curve C there exists an oriented surface S with $\partial S = C$

C

find S!

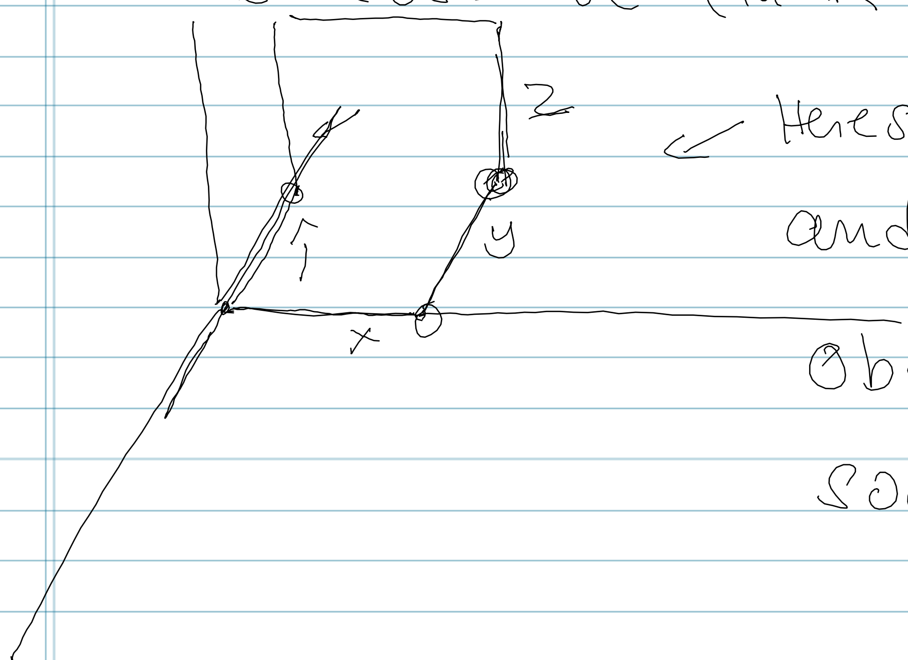
By Stokes thm $\displaystyle\int_C F\cdot T\,ds = \int_S \overbrace{\mathrm{curl}(F)}^{=0}\cdot n\,dS = 0$

Remark The proof that

$F = \nabla f$ Per

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \int_\gamma F \cdot T \, ds \qquad \gamma(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \gamma(1) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is the same as above (in $\mathbb{R}^2$)



Here surface is easy and can be obtained by soap

$\square$

Similar problem

Given F How do I know that $F = \text{curl}(W)$

<u>Answer</u>: iff $\text{div}\, F = 0$ $\quad (\alpha)$

Helmholz Decomposition

F vectorfield, twice differentiable and 0 at ∞.

$$F = -\nabla\phi + \text{curl}(A)$$

$$\phi(x) = \frac{1}{4\pi}\int \frac{\text{div}(F)(y)}{|x-y|}\, dy$$

$$A(x) = \frac{1}{4\pi}\int \frac{\text{curl}(F)(y)}{|x-y|}\, dy$$

<u>lemma</u> $\nabla \times (\nabla \times F) = \text{div}(F) - \Delta F$

(check)

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Proof of Helmholtz

$$A(x) = \frac{1}{4\pi}\int \operatorname{curl}(F)(x+y)\,\frac{dy}{|y|}$$

$\operatorname{curl}(A)$

$$= \frac{1}{4\pi}\int \operatorname{curl}\operatorname{curl}(F)(x+y)\,\frac{dy}{|y|}$$

$$= \frac{1}{4\pi}\int \nabla \operatorname{div}(F)(x+y) - \Delta F(x+y)\,\frac{dy}{|y|}$$

$$-\nabla\Phi + \operatorname{curl}(A)$$

$$= -\frac{1}{4\pi}\int \frac{\operatorname{div}(F)(x+y)}{|y|} + \frac{1}{4\pi}\int \frac{\operatorname{div}(F)(\ldots)}{|y|}$$

$$-\frac{1}{4\pi}\int \Delta F(x+y)\,\frac{dy}{|y|}$$ ← property of Δ

$$= F(x) \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Proved in connection with Maxwell equations. ∇

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Answer (Pb) (α)

$\operatorname{div} F = 0$

+ Helmholtz

$F = \operatorname{curl}(A)$

$\boxed{\text{HELMHOLTZ}}$

Answer

$$\boxed{A = \frac{1}{4\pi} \int \frac{\operatorname{curl} F}{|x-y|}\, dy}$$

Example $\begin{pmatrix} x \\ y \\ -2z \end{pmatrix} \triangleq x\, dy \wedge dz + y\, dz \wedge dx + 2z\, dx \wedge dy$

$\psi = \begin{pmatrix} 2yz \\ 0 \\ xy \end{pmatrix}$ or $\begin{pmatrix} 3yz \\ xz \\ 2xy \end{pmatrix}$

$w = a\, x\, dy\, dz + b\, yz\, dx + c\, xz\, dy$

$dw = a\, y\, \boxed{dx \wedge dz} + a\, \cancel{x}\, dy \wedge dz$

$+ b\, y\, \boxed{dz \wedge dx} + b\, z\, dy \wedge dx$

$+ c\, z\, dx \wedge dy + c\, x\, dz \wedge dy$

$= (cz - bz)\quad dx \wedge dy \qquad c - b = -2$

$+ (by - ay)\quad dz \wedge dx \qquad b - a = 1$

$+ (ax - cx)\quad dy \wedge dx \qquad a - c = 1$

Solutu $a = 1,\ c = 0,\ b = 2 \qquad a = 2\ \ c = 1\ \ b = 3$

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Proof of (+) Assume $f(x) = 0$ $|x| \geq \bar{R}$

Lemma

$$f(0) = \frac{1}{4\pi} \int_{\mathbb{R}^4} (-\Delta) f(x) \frac{dx}{|x|} = RHS$$

$$\int_0^\varepsilon r^2 \frac{dr}{r} = \frac{\varepsilon^2}{2}$$

Pf RHS

$$= \frac{1}{4\pi} \int_{B(0,\varepsilon)} -\Delta f \frac{dx}{|x|} + \frac{1}{4\pi} \int_{B(\varepsilon,R)} f \cancel{\Delta}\left(\frac{1}{|x|}\right) - \frac{\Delta f}{|x|} \, dx$$

$\nabla \xrightarrow{\varepsilon \to 0} 0$

$$= \frac{1}{4\pi} \int_{\partial B(\varepsilon,R)} \left(f \nabla\left(\frac{1}{|x|}\right) - \nabla f \frac{1}{|x|} \right) \cdot n \, dS$$

$\approx \frac{\varepsilon^2}{\varepsilon} \to 0$

$$= \frac{1}{4\pi} - \int_{\partial B(0,\varepsilon)} f \nabla\left(\frac{1}{|x|}\right) \cdot n \, dS + \int_{B(0,\varepsilon)} \frac{\nabla f}{\varepsilon} \cdot n \, dS$$

$$= \frac{1}{4\pi} \int_{\partial B(0,\varepsilon)} f \underbrace{\boxed{\frac{x}{|x|^3} \cdot n}}_{} \, dS \qquad \overline{\boxed{}}^{\,= \frac{1}{\varepsilon^2}}$$

$$= \frac{4\pi\varepsilon^2}{4\pi\varepsilon^2} f(0) \quad \text{by continuity of } f \quad \boxtimes$$

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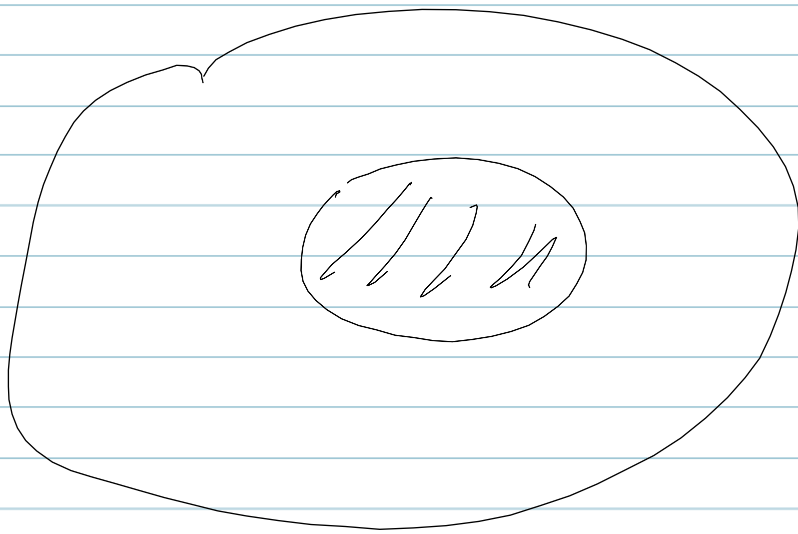
Additional remark

In $\mathbb{R}^2$ $\vec{F}$ conservative on domain D

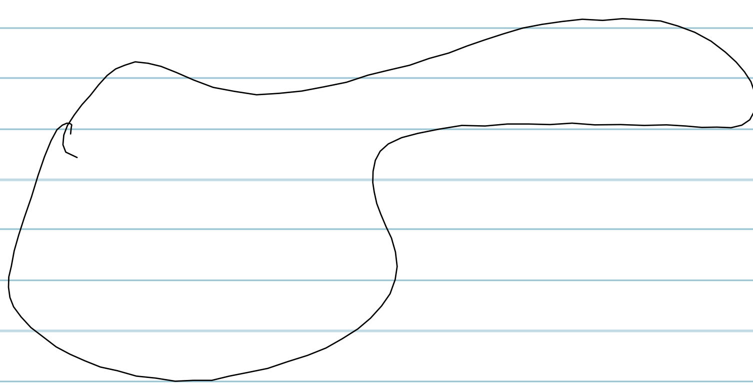
if and only if $\text{curl}(\vec{F}) = 0$

provided that D is simply connected

$\|$ no holes



not simply connect



simply conn

In $\mathbb{R}^3$ simply connected is more difficult!

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