General Stokes theorem

<u>Thm</u> Let M be an oriented manifold. Then

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

<u>Ex</u> $\int_a^b f'(x)\,dx = f(b) - f(a) \underset{\text{Def}}{=} \int_{\partial[a,b]} f$

Ex $\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$

<u>Differential forms</u>

lemma Let $n \in \mathbb{N}$. Then there are matrices V_j ($2^n \times 2^n$) such that

$V_j V_i = -V_i V_j \quad (i \neq j) \qquad V_i^2 = 0$

Ex $n=2$ $V_1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Def $A \subset \{1, \ldots, n\}$ be given by $i_1 < i_2 < \cdots < i_k$.

$$V_A = V_{i_1} V_{i_2} \cdots V_{i_k}.$$

Remark
$$V_j V_A = \begin{cases} 0 & j \in A \\ \pm V_{A \cup \{j\}} & j \notin A \end{cases}$$

Def A differential form on $\mathbb{R}^n$ of order k is given by the matrix valued function

$$w = \sum_A f_A V_A$$

here f_A are infinitely often differentiable

Def

$$dw = \sum_{j=1}^{n} \sum_A \frac{\partial f_A}{\partial x_j} V_j \cdot V_A$$

Ex $w = x_1 V_1 V_2 + x_2 V_2 V_3 + x_3 V_1 V_4$

$$dw = 0 + 0 + V_3 V_1 V_4 = -V_1 V_3 V_4$$
$$= -V_{\{1 3 4\}}$$

Thm $d(d\omega)=0$

Proof $\omega = f_A V_A$

$$d\omega = \sum_{j=1}^{n} \frac{\partial f_A}{\partial x_j} V_j V_A$$

$$d\,d\omega = \sum_{jk} \frac{\partial^2 f_A}{\partial x_j \partial x_k} V_j V_k V_A$$

$$= \sum_{j<k} \frac{\partial^2 f_A}{\partial x_j \partial x_k} \underbrace{(V_j V_k + V_k V_j)}_{=0} V_A = 0$$

Definition Let $g: \underset{\mathbb{R}^n}{\underset{\cap}{D}} \to \mathbb{R}^n$ be a one to one differentiable function. Let $A \subset \{1 \dots n\}$ be a set of cardinality k. $A = \{i_1 \dots$

Define $g_A : D \to \mathbb{R}^k$ by

$$g_A\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} g_{i_1}(u_1, \dots u_n) \\ g_{i_k}(u_1 \dots u_n) \end{pmatrix}$$

$$\boxed{\int_g f_A V_A = \int_D f(g(x_1 \dots u_n)) \det g_A' \, d(u_1 \dots u_k)}$$

Example: $[a,b] \subseteq \mathbb{R}$ $\gamma\ [a,b] \longrightarrow \mathbb{R}^2$

one to one $\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$

$$w = P v_1 + Q v_2$$

$$\int_{\gamma([a,b])} w = \int_{\gamma[a,b]} P v_1 + \int_{\gamma[a,b]} Q v_2$$

$$= \int_a^b P(\gamma(t)) \frac{d\gamma_1}{dt} dt + \int_a^b Q(\gamma(t)) \frac{d\gamma_2}{dt} dt$$

$$= \int_{\gamma[a,b]} P\, dx_1 + \int Q\, dx_2$$

Example: $g\ [0,\pi] \times [0,2\pi] \longrightarrow \mathbb{R}^3$ R

$$g\begin{pmatrix}\theta \\ \varphi\end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta\cos\varphi \\ R\sin\theta\sin\varphi \end{pmatrix}$$

$$\begin{pmatrix} v_2 v_3 v_1 \\ = v_1 v_2 v_3 \end{pmatrix}$$

$$w = P\, v_1 v_2 + R\, v_2 v_3 + Q\, v_3 v_1$$

$$dw = \frac{\partial P}{\partial x_3} v_3 v_1 v_2 + \frac{\partial R}{\partial x_1} v_1 v_2 v_3 + \frac{\partial Q}{\partial x_2} v_2 v_3 v_1 \quad -1$$

$$v_2 v_1 v_3$$

$$= \left(\frac{\partial R}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial P}{\partial x_3} \right) \quad v_1 v_2 v_3$$

Def let $V = \begin{pmatrix} R(x_1 x_2 x_3) \\ Q(x_1 x_2 x_3) \\ P(x_1 x_2 x_3) \end{pmatrix}$ then

$$\operatorname{div} V = \frac{\partial R}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial P}{\partial x_2}$$

$$\int P\, v_1 v_2 = \int P\left(g\begin{pmatrix}\theta \\ \varphi\end{pmatrix}\right) R^2 \sin^2\theta \sin\varphi \, d\theta\, d\varphi$$

$g(D)$ $\quad D$

$$g_{\{1,2\}}\begin{pmatrix}\theta \\ \varphi\end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta\cos\varphi \end{pmatrix}$$

$$\det g'_{\{1,2\}} = \det \begin{array}{cc} \partial/\partial\theta & \partial/\partial\varphi \\ \begin{pmatrix} R(-\sin\theta) & 0 \\ R\cos\theta\cos\varphi & R\sin\theta(-\sin\varphi) \end{pmatrix} \end{array}$$

$$= R^2 \sin^2\theta \sin\varphi$$

Book $\boxed{\int_{g(R)} P\, dx_1 \wedge dx_2} \boxed{= - \int_{g(R)} P\, dx_2 \wedge dx_1}$

$$\int_{g(R)} Q\, V_3\, V_1 = \int_0^{2\pi}\int_0^{\pi} Q(g(\theta,\varphi))\, R^2 \sin^2\theta \cos\varphi \, d\theta\, d\varphi$$

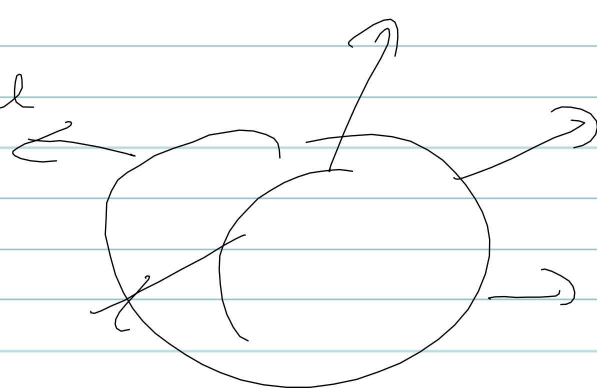
$$g_{(3,1)}\begin{pmatrix}\theta\\ \varphi\end{pmatrix} = \begin{pmatrix} R\sin\theta\sin\varphi \\ R\cos\theta \end{pmatrix}$$

$$g'_{(31)} = \begin{array}{cc} \partial/\partial\theta & \partial/\partial\varphi \end{array}$$
$$\begin{pmatrix} R\cos\theta\sin\varphi & R\sin\theta\cos\varphi \\ R(-\sin\theta) & 0 \end{pmatrix}$$

$$\det = R^2\sin^2\theta\cos\varphi$$

Thm $\boxed{\int_M d\omega = \int_{\partial M} \omega}$

Here parametrization has to point outside



Example

$$w = x\,v_2 v_3 + y\,v_3 v_1 + z\,v_1 v_2$$

$$\boxed{dw = 3\,v_1 v_2 v_3}$$

$$\int_M 3\,v_1 v_2 v_3 = \int_M 3\,d(x_1 x_2 x_3) = 3\,\mathrm{vol}(M)$$

$$3\,\mathrm{vol}(M) = \int_{\partial M} \overset{x_1}{x}\, \cancel{d}x_2 \wedge dx_3 + \int_{\partial M} \overset{x_2}{y}\, dx_3 \wedge dx_1$$

$$+ \int_{\partial M} \overset{x_3}{z}\, dv_1 \wedge dv_2$$

$$M = B_R = \left\{ \begin{pmatrix} r\cos\theta \\ r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \end{pmatrix} : \begin{array}{l} 0 \le r \le R \\ 0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi \end{array} \right\}$$

$$\int_{\partial B_R} x_1\, dx_2 \wedge dx_3 = 2\pi R^3 \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta = \frac{4\pi R^3}{3}$$

$$g_{23}\begin{pmatrix}\theta \\ \varphi\end{pmatrix} = \begin{pmatrix} R\sin\theta\cos\varphi \\ R\sin\theta\sin\varphi \end{pmatrix} \qquad g_{23}'\begin{pmatrix}\theta \\ \varphi\end{pmatrix} = \begin{pmatrix} R(+\cos\theta)\cos\varphi & R s\theta(-s\varphi) \\ R(+\cos\theta)\sin\varphi & R s\theta\, c\varphi \end{pmatrix}$$

$$\det g_{23}'\begin{pmatrix}\theta \\ \varphi\end{pmatrix} = R^2 + \cos\theta\sin\theta\ (\cos^2\varphi + \sin^2\varphi)$$

$$= R^2 \cos\theta\sin\theta$$

$$\int_{\partial M} x_2 \, dx_3 \wedge dx_1 = R^3 \int_0^{2\pi} \int_0^{\pi} \sin^3\theta \cos^2\varphi \, d\theta \, d\varphi$$

$$g_{31}\begin{pmatrix}\theta \\ \varphi\end{pmatrix} = \begin{pmatrix} R \sin\theta \sin\varphi \\ R\cos\theta \end{pmatrix}$$

$$g_{31}'\begin{pmatrix}\theta \\ \varphi\end{pmatrix} = \begin{pmatrix} R\cos\theta \sin\varphi & R\sin\theta\cos\varphi \\ R(-\sin\theta) & 0 \end{pmatrix}$$

$$= R^2 \sin^2\theta \cos\varphi$$

$$= R^3 \pi \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{4}{3}$$

$$\int \sin^2\theta \sin\theta = \sin^2\theta(-\cos\theta) + \int 2\sin\theta\cos\theta\cos\theta$$

$$= -\cos\theta\sin^2\theta + 2\int \sin\theta - 2\int \sin^2\theta)$$

$$\int \sin^3\theta = \frac{1}{3}\left(-\sin^2\theta\cos\theta - 2\cos\theta\right)$$

$$\frac{1}{3}$$

$$\int_{\partial M} x_3 \, dx_1 \wedge dx_2$$

$$g_{12}\begin{pmatrix}\theta\\ \varphi\end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \cancel{\cos\varphi} \end{pmatrix} \qquad g_{12}' = \begin{array}{cc} \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ \end{array} \begin{pmatrix} R(-\sin\theta) & 0 \\ R\cos\theta\sin\varphi & -R\sin\theta\,\cancel{\sin\varphi} \end{pmatrix}$$

$$\det = R^2 + \sin^2\theta \;\cancel{\sin}\varphi \qquad x_3 = R\sin\theta\sin\varphi$$

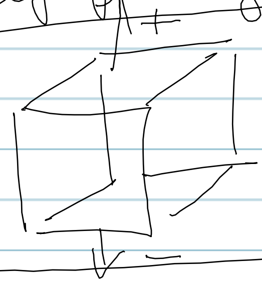
$$= \int_0^{2\pi} \left(\int_0^{\pi} + \sin^3\theta \, d\theta \right) \cancel{\cos} \sin^2\varphi \, d\varphi = R^3 \pi \frac{4}{3}$$

$$\int_0^{2\pi} \cos\varphi\sin\varphi\, d\varphi = \left.\frac{\sin^2\varphi}{2}\right|_0^{2\pi}$$

Hence $\boxed{Vol = \frac{4R^3\pi}{3}}$ Okay,

$\boxed{\text{Proof of divergence}}$

$$\boxed{\omega = P\, \underset{V_1}{dx_1} \wedge \underset{V_2}{dx_2}}$$



$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

Case I $\quad \omega = P\, dx_1 \wedge dx_2$

$$g^1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

$$g^2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$$

$$\int_{\partial M} P\, dx_1 \wedge dx_2 = \int_{[0,1]^2} P\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} dx_1 x_2 - \int P\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} dx_1 dx_2$$

$$= \int_{[0,1]^3} \frac{\partial P}{\partial x_3}(x_1 x_2 x_3)\, d x_1 dx_2 dx_3$$

$$= \int_M d\omega$$

Other cases similar

Short version in $\mathbb{R}^3$

$$\boxed{\int_B \operatorname{div}(V)\, dvol = \int_{\partial D} V \cdot n\, dS}$$