Practice problems -Solutions

1) Recall the definition of linearly independent vectors.

Solutions The vectors $v_1, .., v_m$ are linearly independent if

$$\sum_j \lambda_j v_j \; = \; 0$$

only occurs for $\lambda_1 = \cdots = \lambda_m = 0$. If $v_1, ..., v_m$ are in $\mathbb{R}^k$ then this is the case iff and only if the determinant of the matrix of inner products

$$m_{ij} \; = \; v_i \cdot v_j$$

is non-0.

2) Differentiate

$$f(x) \; = \; \frac{\partial}{\partial x} \int_0^{x^2+y^2} (x^2+y^3)^{1/3} dy \; .$$

Solution: We introduce

$$F(x,z) \; = \; \int_0^z (x^2+y^3)^{1/3} dy$$

and find

$$f(x) \; = \; F(x,x^2)$$

and hence

$$\begin{aligned} f'(x) &= \frac{\partial F}{\partial x}(x,x^2) + \frac{\partial F}{\partial z}(x,x^2)2x \\ &= \int_0^{x^2} 1/3(x^2+y^3)^{-2/3} 2x dy + (x^2+x^6)^{1/3} 2x \\ &= 2x(\int_0^{x^2} 1/3(x^2+y^3)^{-2/3} dy + x^{2/3}(1+x^4)^{1/3}) \; . \end{aligned}$$

Calculate the area of the Archimedean spiral.

Solution: We have $r(\theta) = a\theta$ and hence the boundary of one cycle is given by

$$g(\theta) \; = \; r(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \; = \; a\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \; .$$

We may integrate xdy and deduce from Green (and

$$dy \; = \; d(a\theta \sin\theta) \; = \; a\sin\theta + a\theta\cos\theta)$$

that

$$
\begin{aligned}
area(sp) &= \int_0^{2\pi} a\theta \cos(\theta)(a \sin\theta + a\theta\cos\theta) d\theta \\
&= a^2 \int_0^{2\pi} \theta \sin\theta \cos\theta d\theta + a^2 \int_0^{2\pi} \theta^2 \cos^2(\theta) d\theta \ .
\end{aligned}
$$

Time for integration by parts.

$$
2 \int_0^{2\pi} \theta \sin\theta \cos\theta d\theta = \theta \sin^2\theta|_0^{2\pi} - \int_0^{2\pi} \sin^2\theta d\theta \ = \ -\pi \ .
$$

Moreover,

$$
\begin{aligned}
\int \cos^2\theta d\theta &= \int \cos\theta \cos\theta d\theta \\
&= \sin\theta\cos\theta + \int \sin\theta \sin\theta \\
&= \sin\theta\cos\theta + \theta - \int \cos^2\theta \ .
\end{aligned}
$$

Hence

$$
\int \cos^2\theta d\theta \ = \ \frac{\sin\theta\cos\theta + \theta}{2} \ .
$$

This gives

$$
\begin{aligned}
\int_0^{2\pi} \theta^2 \cos^2(\theta) d\theta &= \frac{\sin\theta\cos\theta + \theta}{2} \theta^2|_0^{2\pi} - \int 2\theta \frac{\sin\theta\cos\theta + \theta}{2} \\
&= \frac{(2\pi)^3}{2} - \int_0^{2\pi} \theta^2 d\theta - \int \theta \sin\theta\cos\theta d\theta \\
&= \frac{(2\pi)^3}{6} - \int_0^{2\pi} \theta \sin\theta\cos\theta d\theta \\
&= \frac{(2\pi)^3}{6} + \frac{\pi}{2} \ .
\end{aligned}
$$

Thus we get

$$
a^2(\frac{(2\pi)^3}{6}) \ .
$$

3)

1. Calculate the flux

$$
\int_G F \cdot n dS \ .
$$

a) $F(x_1, x_2, x_3) = \begin{pmatrix} x_1^2 x_2 \\ x_2^2 x_3 \\ x_3^2 x_1 \end{pmatrix}$ and G is the upper hemisphere with radius 1.

b) $G = \left\{ \begin{pmatrix} y_1^3 - y_2 \\ y_2^3 - y_3 \\ y_3 \end{pmatrix} : y_1^2 + y_2^2 + y_3^3 = 1 \right\}$ and $F(x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$.
You may use $\int_0^{2\pi} \cos^4(\phi) d\phi = \frac{3\pi}{4}$ and $\int_0^{\pi} \sin^4 \theta d\theta = \frac{3\pi}{8}$.

Solution: Let K the half ball. Then we note that

$$F \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = x_3^2 x_1 = 0 \,.$$

Thus, by the divergence theorem

$$\int_G F \cdot n dS = \int_K (2x_1x_2 + 2x_2x_3 + 2x_3x_1) d(x_1, x_2, x_3) = 0 \,.$$

Indeed, we may change x_1 to $-x_1$ without changing the body. Hence the first and the third term are 0. We repeat this for x_2 and get the result. For part b) we change in into a differential form

$$\omega = x_1 dx_2 \wedge dx_3$$

and use the change of variable

$$x_1 = y_1^3 - y_2 \, x_2 = y_2^3 - y_3 \, x_3 = y_3 \,.$$

This gives $dx_2 = 3y_2^2 dy_2 - dy_3$, $dy_3 = dx_3$ and hence

$$x_1 dx_2 \wedge dx_3 = (y_1^3 - y_2) dy_2 \wedge dy_3$$

Now we have to use the standard parametrization of the sphere $g(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$ with

$$\det g'_{23} = \sin^2\theta \cos\phi \,.$$

This gives

$$\begin{aligned}
\int_G F \cdot ndS &= \int_{y_1^2+y^2+y_3^2=1} (y_1^3 - y_2) dy_2 \wedge dy_3 \\
&= \int_0^{2\pi} \int_0^{\pi} (\sin^3\theta \cos^3\phi - \sin\theta \sin\phi) \sin^2\theta \cos\phi d\theta d\phi \\
&= \int_0^{2\pi} \int_0^{\pi} \sin^4\theta \cos^4\phi d\theta d\phi \\
&= \int_0^{2\pi} \cos^4\phi d\phi \int_0^4 \sin^4\theta d\theta \ .
\end{aligned}$$

By the hint we get $9/32\pi^2$.

6. Derive the divergence theorem from the general Stokes theorem.

Solution: Let

$$\omega \ = \ F_1 dx_2 \wedge dx_3 + F_2 dx_3 \wedge dx_1 + F_3 dx_1 \wedge x_2$$

be a differential from of degree 2. Then

$$d\omega \ = \ (\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}) dx_1 \wedge dx_2 \wedge dx_3 \ .$$

By the general Stokes theorem we have

$$\int_K \mathrm{div}(F) d\mathrm{vol} \ = \ \int_K d\omega \ = \ \int_{\partial K} \omega$$

provided the boundary is outward oriented. Let $g : D \to \mathbb{R}^3$ be a parametrization with coordinate function g_1, g_2, g_3. Then we observe that

$$\frac{\partial g}{\partial s} \times \frac{\partial g}{\partial t} \ = \ \begin{pmatrix} \frac{\partial g_2}{\partial s} - \frac{\partial g_3}{\partial t} \\ \frac{\partial g_3}{\partial s} - \frac{\partial g_1}{\partial t} \\ \frac{\partial g_1}{\partial 2} - \frac{\partial g_2}{\partial t} \end{pmatrix} \ = \ \begin{pmatrix} \det g'_{23} \\ \det g'_{31} \\ \det g'_{12} \end{pmatrix} \ .$$

Thus by the definition we deduce that indeed

$$\int_{\partial K} d\omega \ = \ \int_D F(\gamma(s,t)) \cdot N(s,t) dA(s,t) \ = \ \int F \cdot ndS \ .$$

7. Calculate in two different ways

$$\int_{-1 \le x_1 \le 1, 0 \le x_2 \le 1 - x_1^4, x_3 = 1 - x_1^4 - x_2} x_1^3 dx_2 \wedge dx_3 \ .$$

Solution: We use $\omega = x_1^3 dx_2 \wedge dx_3$ and

$$d\omega \;=\; 3x_1^2 dx_1 \wedge x_2 \wedge x_3 \,.$$

By the divergence theorem we deduce for

$$K \;=\; \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;:\; -1 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1^4, 0 \leq x_3 \leq 1 - x_1^4 - x_2 \}$$

that

$$\begin{aligned}
\int_K d\omega \;&=\; \int_{-1}^{1} \int_0^{1-x_1^4} \int_0^{1-x_1^4-x_2} 3x_1^2 dx_3 dx_2 dx_1 \\
&= 3 \int_{-1}^{1} \int_0^{1-x_1^4} (1 - x_1^4 - x_2) dx_2 x_1^2 dx_1 \\
&= 3 \int_{-1}^{1} (1 - x_1^4)^2 - \frac{(1 - x_1^4)^2}{2} x_1^2 dx_1 \\
&= \frac{3}{2} \int_{-1}^{1} (1 - x_1^4)^2 x_1^2 dx_1 \\
&= 3 \int_0^1 (1 - 2x_1^4 + x_1^8) x_1^2 dx_1 \\
&= 3(\frac{1}{3} - \frac{2}{7} + \frac{1}{11}) \;=\; \frac{77 - 66 + 21}{77} \;=\; \frac{32}{77} \,.
\end{aligned}$$

However, we have to be careful with two other pieces of the boundary. On in front in the x_1, x_3 plane. In the x_2, x_3 plane we have $x_1 = 0$ and hence

$$\int_K d\omega \;=\; \int_{\partial K_f} \omega \;=\; \int_{-1 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1^4, x_3 = 1 - x_1^4 - x_2} x_1^2 dx_2 \wedge dx_3 \,.$$

The other possibility is the definition. We use $h(x_1, x_2) = 1 - x_1^4 - x_2$ and

$$g(x_1, x_2) \;=\; \begin{pmatrix} x_1 \\ x_2 \\ h(x_1, x_2) \end{pmatrix} .$$

We get

$$\det g'_{23} \;=\; \det \begin{pmatrix} 0 & 1 \\ \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{pmatrix} \;=\; -\frac{\partial h}{\partial x_1} \;=\; 4x_1^3 \,.$$

This gives

$$\int_{-1\le x_1\le 1, 0\le x_2\le 1-x_1^4, x_3=1-x_1^4-x_2} x_1^3 dx_2\wedge dx_3 \;=\; \int_{-1\le x_1\le 1, 0\le x_2\le 1-x_1^4} x_1^3(4x_1^3)dx_2dx_1$$

$$= 4\int_{-1}^{1}(1-x_1^4)x_1^6 dx_1$$

$$= 8\int_0^1 (x_1^6 - x_1^{10})dx_1 \;=\; 8(\frac{1}{7}-\frac{1}{11}) \;=\; 8(\frac{11-7}{77}) \;=\; \frac{32}{77}\,.$$

Pretty cool.

8. 5) Let

$$G \;=\; \left\{ \begin{pmatrix} y_1 - y_2^2 \\ y_2 \\ y_3 \end{pmatrix} : y_1^2+y_2^2+y_3^2 = 1, y_1 \;\ge\; 0 \right\}$$

and $F(x_1,x_2,x_3) \;=\; \begin{pmatrix} x_3 \\ 0 \\ 0 \end{pmatrix}$. Calculate in two different ways

$$\int_G \mathrm{curl}(F)\cdot n dS\,.$$

Solution: Let $\omega = x_3 dx_1$ and

$$x_1 = y_1 - y_2^2, x_2 = y_2, x_2 = y_3$$

the change of variables. We get

$$\omega \;=\; y_3 d(y_1 - y_2^2) \;=\; y_3 dy_1 - 2y_3y_2dy_2$$

Then Stokes theorem (together with $dy_1 = 0 = y_1$, $y_2 = \cos\theta$, $y_3 = \sin\theta$) gives

$$\int_G \mathrm{curl}(F)\cdot n dS = \int_{\partial G} \omega\cdot T ds$$
$$= \int_0^{2\pi} -2\sin\theta\cos\theta(-\sin\theta)d\theta \;=\; 0\,.$$

As for the surface integral we have

$$d\omega \;=\; dx_3\wedge dx_1 \;=\; dy_3\wedge dy_1 - 2y_2 dy_3\wedge dy_2\,.$$

We use $g(\theta,\phi) = \begin{pmatrix} \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{pmatrix}$ with

$$g'(\theta,\phi) = \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta\cos\phi & \sin\theta(-\sin\phi) \\ \cos\theta\sin\phi & \sin\theta\cos\phi \end{pmatrix} .$$

and get

$$\det g'_{31} = \sin^2\theta\cos\phi \, , \; \det g'_{23} = \cos\theta\sin\theta \, .$$

This gives $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq 2\pi$,

$$\int_G d\omega = \int_0^{\pi/2} \int_0^{2\pi} [\sin^2\theta\cos\phi + 2\sin\theta\cos\phi\cos\theta\sin\theta] d\phi d\theta = 0 \, .$$

9. Decide whether the following vector fields are conservative

a) $F(x_1,x_2) = \begin{pmatrix} x_1^2 x_2^3 \\ x_1^3 x_2^2 \end{pmatrix}$.

b) $F(x_1,x_2,x_3) = \begin{pmatrix} \cos(x_1^2x_2^2x_3^2)2x_1 \\ \cos(x_1^2x_2^2x_3^2)2x_2 \\ \cos(x_1^2x_2^2x_3^2)2x_3 \end{pmatrix}$.

Solution: In a) we have to $\frac{\partial Q}{\partial x_1} = \frac{\partial P}{\partial x_2}$. In our case

$$\frac{\partial Q}{\partial x_1} = 3x_1^2x_2^2 = \frac{\partial P}{\partial x_2} \, .$$

For Problem b) we have to check three of these:

$$\mathrm{curl}(F) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \times \begin{pmatrix} \cos(x_1^2x_2^2x_3^2)2x_1 \\ \cos(x_1^2x_2^2x_3^2)2x_2 \\ \cos(x_1^2x_2^2x_3^2)2x_3 \end{pmatrix}$$
$$= \begin{pmatrix} (-4)x_2x_3\sin(x_1^2x_2^2x_3^2) - (-4)(x_2x_3)\sin(x_1^2x_2^2x_3^2) \\ (-4)x_1x_3\sin(x_1^2x_2^2x_3^2) - (-4)x_1x_3\sin(x_1^2x_2^2x_3^2) \\ (-4)x_1x_2\sin(x_1^2x_2^2x_3^2) - (-4)x_1x_2\sin(x_1^2x_2^2x_3^2) \end{pmatrix}$$

which of course is 0.