The spanning problem in 2D

<u>Problem I</u> Given two vectors v, w in $\mathbb{R}^2$.

when is it true that for all z in $\mathbb{R}^2$

we can find λ, μ such that

$$(*) \quad z = \lambda v + \mu w \quad ?$$

Examples 1) $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

2) $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3) $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $w = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

4) $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $w = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

Answer: If v, w do not point in same direction up to sign !

Def $\{v, w\}$ is called a basis for $\mathbb{R}^2$ if every point can be expressed by $(*)$

Problem II Let $\{v, w\}$ be a basis.

How can we find λ, μ ?

$\not\# \; z = \begin{pmatrix} x \\ y \end{pmatrix}$

Answer: Solve linear equation $v = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ $w = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$

$$x = \lambda a_{11} + \mu a_{21}$$

$$y = \lambda a_{21} + \mu a_{22}$$

Examples

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\lambda + \mu = 3$$
$$2\lambda + 3\mu = 7$$

$\mu = 3 - \lambda$

$$7 = 2\lambda + 3(3-\lambda)$$
$$= 9 - \lambda$$

$\lambda = 2 \quad \mu = 1.$

General method

Definition A $n \times m$ matrix is a collection

1) of $n \cdot m$ real numbers

$$A = \begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \\ & & & \\ a_{n1} & & & a_{nm} \end{bmatrix}$$

2) Let $x = \begin{pmatrix} x_1 \\ \\ x_m \end{pmatrix}$ be a vector in m space.

Then $Ax = \begin{bmatrix} a_{11}x_1 + \cdots & + a_{1m}x_m \\ & \\ a_{n1}x_1 + \cdots & a_{nm}x_m \end{bmatrix}$

is a vector in n-space

3) One can multiply a $n \times m$ matrix with a $m \times k$ matrix and gets a $n \times k$ matrix

$$A \cdot B = \begin{bmatrix} a_{12}b_{11} + \cdots + a_{1m}b_{m1} & & & \\ & \vdots & \cdot & \cdot \\ a_{n1}b_{11} + \cdots a_{nm}b_{m1} & & a_{n1}b_{m1} & a_{nm}b_{m1} \end{bmatrix}$$

How helpful?

<u>Observation:</u>

$$z = \lambda v + \mu w$$

means $z = a\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ $\qquad a = \begin{pmatrix}\downarrow & \downarrow\end{pmatrix}$

If we can find c such that

$$ca = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ then:}$$

$$c(z) = ca\begin{pmatrix}\lambda\\ \mu\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda\\ \mu\end{pmatrix} = \begin{pmatrix}\lambda\\ \mu\end{pmatrix}$$

Hence λ, μ are given by $c(z)$!

We traded ~~...~~ Problem II against

<u>Problem III</u> Find c with $ca = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$!

<u>Note</u> Pb II and Pb III are really equivalent (equally difficult) if we look for a general algorithm which works for all z (HW)

Key Example

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $0 \cdot f + ch = 1 \qquad h = \frac{1}{c}$

$\underbrace{0 \cdot e}_{=0} + cg = 0 \qquad g = 0$

$ae = 1 \qquad e = \frac{1}{a}$

$af + bh = 0$

$f = -\frac{bh}{a} = -\frac{b}{ac}$

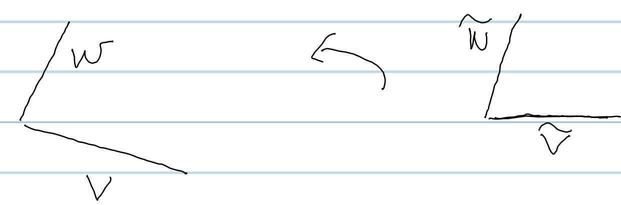
Solution is

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{a^{-1} = \frac{1}{ac} \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}}$$

The inverse of a

Q General case?



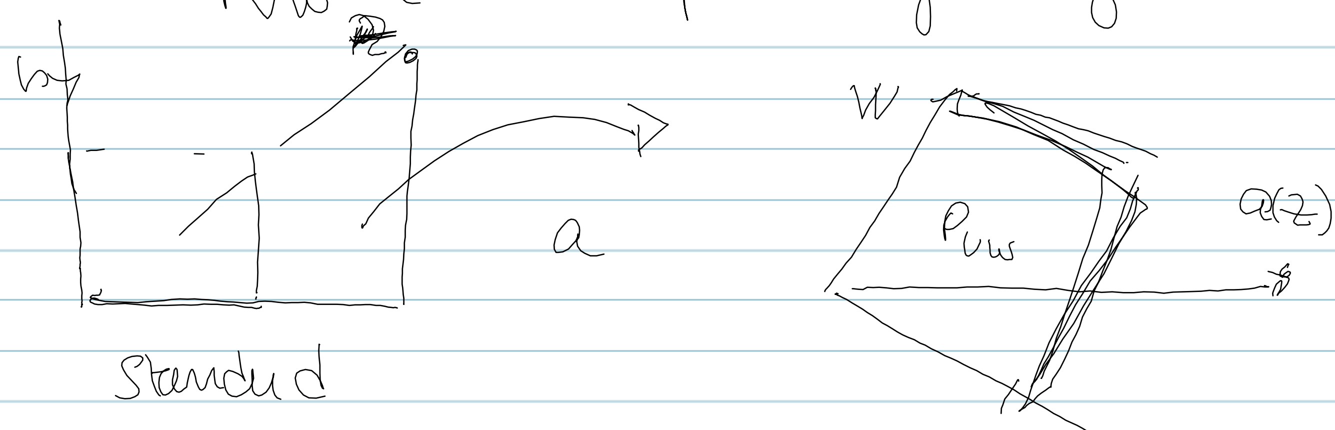
$\tilde{v} = (a, 0)$

New matrix $\quad m = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$

$\hat{m} = \begin{pmatrix} \hat{v}_1 & \hat{w}_1 \\ 0 & \hat{w}_2 \end{pmatrix} \quad \leftarrow$ Here we know what to do.

Determinants and area

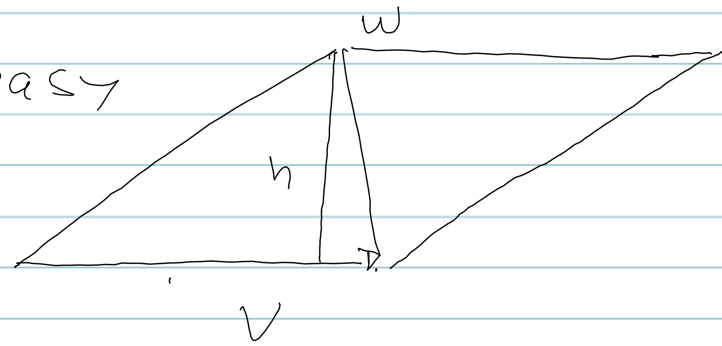
Pb IV Let $\{v, w\}$ be a basis and P_{vw} be the unit parallelogram given



Standard Coordinates

Again for $v = \begin{pmatrix}\alpha \\ 0\end{pmatrix}$ $w = \begin{pmatrix}\beta \\ \gamma\end{pmatrix}$

this is easy



Area $= 2\, b\, \frac{\text{height}}{2} = \text{base height}$

$= \alpha\beta$

<u>Theorem</u> The area of P_{wv} is given by ~~the~~

$|a_{11}a_{22} - a_{12}a_{21}|$

provided $v = \begin{pmatrix}a_{11} \\ a_{21}\end{pmatrix}$ $W = \begin{pmatrix}a_{12} \\ a_{22}\end{pmatrix}$

(This means

$v = a\begin{pmatrix}1 \\ 0\end{pmatrix}$ $w = a\begin{pmatrix}0 \\ 1\end{pmatrix}$

$\begin{pmatrix}1 \\ 0\end{pmatrix}$ $\begin{pmatrix}0 \\ 1\end{pmatrix}$ are called standard unit vectors

Defenition: $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 matrix. Then the determinant of a is given by

$$\det(a) = a_{11}a_{22} - a_{12}a_{21}$$

(This mean $\text{Area}(P_{w,v}) = |\det(a)|$)

<u>Lemma</u> $\det(ab) = \det(a)\det(b)$

<u>Proof</u> $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$\det(a)\det(b) = (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$$

$$= a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}$$

$$- a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22}$$

$$ab = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$\det(ab) = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$$

$$= a_{11}a_{21}b_{11}b_{12} + a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{21}b_{12} + a_{12}b_{21}a_{22}b_{22}$$
$$- a_{11}a_{21}b_{12}b_{11} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{22}b_{11} - a_{12}a_{22}b_{22}b_{21}$$

$$= a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{21}b_{12} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22}$$

$$= \det(a)\det(b)$$ ∎

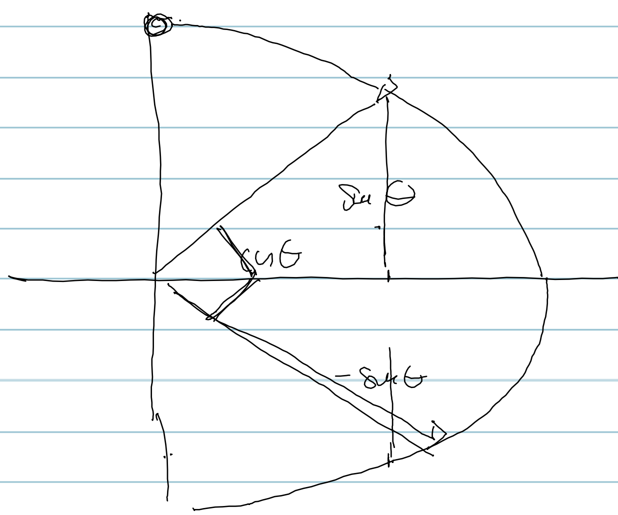
Fact The matrix of a clockwise rotation is given by

$$U_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and satisfies

$$\det(U_\theta) = 1$$

$$U_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} \qquad U_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$



Proof of Thm: Let v, w be given.

Choose u_Θ such that

$\tilde{v}$ $\quad u_\Theta(v) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \qquad u_\Theta w = \tilde{w}$

Then the new matrix is $\tilde{a} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ and area is $|\alpha\gamma| = |\det \tilde{a}|$

$$\text{area}(P_{v,w}) = \text{area } P_{\tilde{v},\tilde{w}}$$
$$= |\det \tilde{a}|$$
$$= |\det u_\Theta a|$$
$$= \underbrace{|\det u_\Theta|}_{=1} \, |\det a|$$
$$= |\det a| \qquad \blacksquare$$

Theorem: Let $a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then

$$a^{-1} = \frac{1}{\det a} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

Proof

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \alpha\delta - \beta\gamma & \alpha\beta - \beta\alpha \\ \gamma\delta - \delta\gamma & \alpha\delta - \beta\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \det(a) & 0 \\ 0 & \det(a) \end{pmatrix} \qquad \blacksquare$$