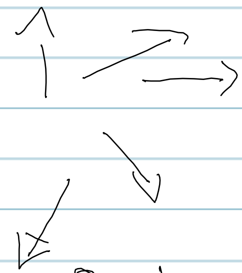
Some problems in $\mathbb{R}^2$ (2D)

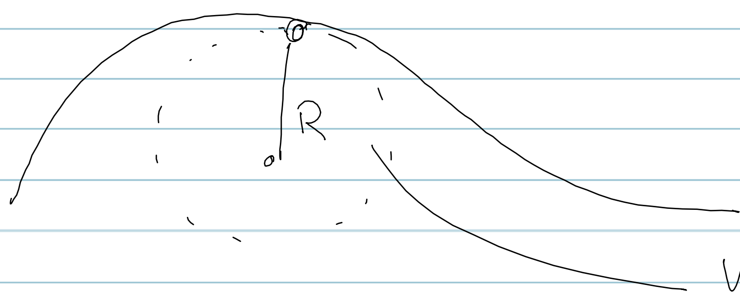
Pb let  be a vector fied in $\mathbb{R}^2$

let P_0, P_1 be a points in $\mathbb{R}^2$

a) Find a curve which connects p_0 and p_1 which follows the flow

b) Detemine the length of the path.

c) Fnd the smallest curvature of path.



P path

maxmal radius inscribed curve

Curvature $= \frac{1}{\max \text{Radius}}$

Derivatives

$p(t) = (x(t), y(t))$

$\frac{dp}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right)$ velocity at point $p(t)$

$\frac{d^2p}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right)$ "acceleration"

Example $V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ vector field

$p_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ Find

$\frac{dp}{dt} = V(p(t))$

This means

$\frac{dx}{dt} = x \qquad x(0) = x_0$

$\frac{dy}{dt} = y \qquad y(0) = y_0$

Solution $x(t) = x_0 e^t \quad y(t) = y_0 e^t$

Of course we can only move along

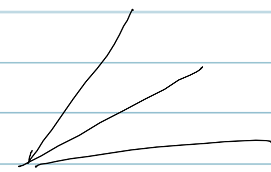
lines ~~with 45~~ degrees

(curvature = 0)

$x_0 = 1 \quad y_0 = 2$

$p(t) = (e^t, 2e^t)$

New variable x instead of t.



$$\boxed{p(x) = (x, 2x)}$$

Example $V\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-y\\x\end{pmatrix}$

$$\frac{dp}{dt} = V(p(t))$$

$$\frac{dx}{dt} = -y \qquad \text{Then} \quad \frac{d^2x}{dt^2} = -x$$

$$\frac{dy}{dt} = x$$

Solution $x(t) = a\cos t + b\sin t$

$y(t) = -a\sin t + b\cos t$

$$\boxed{x(0) = a \quad y(0) = b}$$

Rules for Derivatives

1) $$\frac{dp}{dt} + \frac{dq}{dt} = \frac{d(p+q)}{dt}$$

2) If $t = g(s)$ Then

$$\frac{dp}{ds} = \frac{dp}{dt}\,\frac{dt}{ds} \qquad \text{Product rule}$$

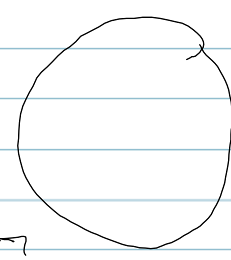
$$\frac{d^2p}{ds^2} = \frac{d}{ds}\left(\frac{dx}{dt}\,\frac{dt}{ds},\ \frac{dy}{dt}\,\frac{dt}{ds} \right)$$

$$= \left(\frac{d^2t}{ds^2}\,\frac{dx}{dt} + \frac{d^2x}{dt^2}\left(\frac{dt}{ds}\right)^2,\ \frac{dy}{dt}\,\frac{d^2t}{ds^2} + \frac{d^2y}{dt^2}\left(\frac{dt}{ds}\right)^2 \right)$$

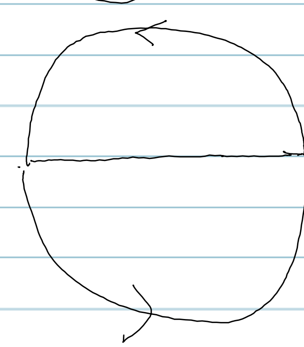
Example

$x(t) = \cos t \qquad y = \sin t$

$p(t) = (\cos t, \sin t)$



"$s = x$" $\qquad x = \cos t \qquad \sin t = \pm\sqrt{1 - x^2}$



$q_{\pm}(x) = (x, \pm\sqrt{1-x^2})$

Need two curves

and $-1 \le x \le 1$

$$\frac{dp}{dt} = (-\sin t, \cos t)$$

$$\boxed{\frac{dq_+}{dx} = \left(1, \pm \frac{-x}{\sqrt{1-x^2}}\right)}$$