Reparametrization by arc length.

p given

$$\int_0^t \left|\frac{dp}{dt}\right| dt = s$$

<u>Theorem</u> a) $\left|\frac{dp}{ds}\right| = 1$

$$\frac{dp}{ds} \perp \frac{d^2p}{ds^2}$$

<u>Proof</u> a) $\frac{dp}{ds} = \frac{dp}{dt}\frac{dt}{ds}$

$$\frac{ds}{dt} = \left|\frac{dp}{dt}\right| \qquad \frac{dt}{ds} = \frac{1}{\left|\frac{dp}{dt}\right|}$$

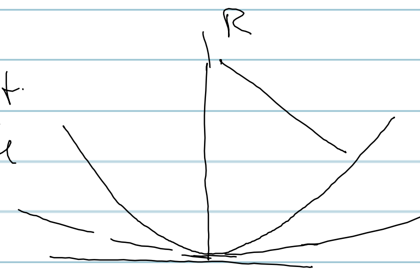
$$\left|\frac{dp}{ds}\right| = \left|\frac{dt}{ds}\right| \left|\frac{dp}{dt}\right| = \frac{\left|\frac{dp}{dt}\right|}{\left|\frac{dp}{dt}\right|} = 1$$

b) $0 = \frac{d}{ds} 1 = \frac{d}{ds}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right)$

$$= 2\frac{dx}{ds}\frac{d^2x}{ds^2} + 2\frac{dy}{ds}\frac{d^2y}{ds^2}$$

$$= 2\left(\frac{dx}{ds}, \frac{dy}{ds}\right) \cdot \left(\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}\right)$$

Lemma $f(0)=0$ and f twice cont. differentiable



$$0 \le f(x) \le R - \sqrt{R^2 - x^2}$$

Then $f'(0) = 0$ and

$$0 \le f''(0) \le \frac{1}{R}$$

Proof: $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{f(x)}{x}$

However for $x > 0$

$0 \le \dfrac{f(x)}{x}$; Hence $f'(0) \ge 0$

Moreover

$$0 \le \frac{f(x)}{x} \le \frac{R - \sqrt{R^2 - x^2}}{x} = \frac{R^2 - (R^2 - x^2)}{x(R + \sqrt{R^2 - x^2})}$$

$$= \frac{x}{R + \sqrt{R^2 - x^2}} \xrightarrow[x \to 0]{} 0$$

Hence $f'(0) \le 0$

Thus $f'(0) = 0$

For the second part we use Taylor formula.

$$f(x) = f(0) + f'(0)x + \int_0^x f''(r)(x-r)\,dr$$

$$= \int_0^x f''(r)(x-r)\,dr.$$

For x small enough

$|f''(r) - f''(0)| < \varepsilon$ small mister.

$$\int_0^x f''(r)(x-r) \le \int_0^x f''(0)(x-r)\,dr + \varepsilon \int_0^x (x-r)\,dr$$

$$= f''(0)\frac{x^2}{2} + \varepsilon \frac{x^2}{2}$$

Similar

$$\int_0^x f''(r)(x-r) \geq f''(0)\frac{x^2}{2} - \varepsilon \frac{x^2}{2}$$

By assumption

$$0 \le f(x) = \int_0^x f''(r)(x-r)\,dr \le (f''(0)+\varepsilon)\frac{x^2}{2}$$

Hence

$$0 \le f''(0) + \varepsilon \qquad (\varepsilon \text{ small} \Rightarrow f''(0) \geq 0)$$

Similar:

$$-\varepsilon\frac{x^2}{2} + f''(0)\frac{x^2}{2} \le f(x) \le R - \sqrt{R^2-x^2}$$

$$= \frac{x^2}{R+\sqrt{R^2-x^2}}$$

Therefore

$$-\varepsilon + f''(0) \leq \frac{1}{R+\sqrt{R^2-x^2}}$$

We send x to 0 and get

$$\frac{-\varepsilon + f''(0)}{2} \leq \frac{1}{2R}$$

(ε small)

$$\frac{f''(0)}{2} \leq \frac{1}{2R}$$

Thus

$$0 \leq f''(0) \leq \frac{1}{R} \qquad \boxed{\Xi}$$

Conclusion: The biggest possible R is

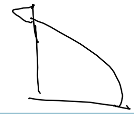
$$\boxed{R = \frac{1}{f''(0)}} \qquad \boxed{\text{curvature} = \frac{1}{R} = y''(0)}$$

Theorem: p parametrized according to arclength.

Then $\kappa = \frac{d^2p}{ds^2} \cdot \left(\frac{dp}{ds}\right)^{\perp}$ ← the orthogonal vector !

$$= \left|\frac{d^2p}{ds^2}\right|$$

Definition $(x,y)^{\perp} = (-y, x)$

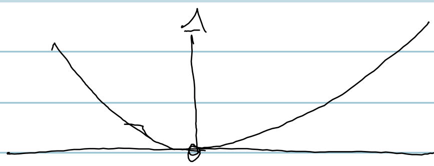


Proof We may assume $p(0) = 0$

$p'(s) = (1, 0)$

$p''(s) = (0, \alpha)$

Then $(0,1)$



O

$\frac{d^2y}{ds^2} = \alpha$

$$\left(\begin{aligned} \text{Curvature} &= p''(s) \cdot (0,1) \\ &= p''(s) \cdot p'(s)^{\perp} \end{aligned} \right)$$

□

$$\boxed{\kappa = \frac{d^2p}{ds^2} = \left|\frac{d^2p}{ds^2}\right|}$$

Let $T(t) = \dfrac{\frac{dp}{dt}}{\left|\frac{dp}{dt}\right|} = \frac{dp}{ds}$

Then $\kappa = \left|\frac{dT}{ds}\right|$. By chain rule

$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt}$ Hence

Recap

$$\boxed{\kappa = \left|\frac{dT}{ds}\right| = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{ds}{dt}\right|}} = \frac{|T'(t)|}{|r'(t)|} \quad \text{(book page 851)}$$

$\square$