Properties of linear maps

— maps given by matrix multiplication.

a $n \times m$ matrix. The product

$$a \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1m}x_m \\ \\ a_{n1} + \cdots + a_{nm}x_m \end{pmatrix}$$

gives a map $T_a : \mathbb{R}^m \to \mathbb{R}^n$.

This map has the following properties

1) $T_a \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\boxed{\text{ii)} \quad T_a(\lambda v + \mu w) = \lambda T_a(v) + \mu T_a(w)}$$

Maps with second property are called linear maps

$T_a \in L(\mathbb{R}^m, \mathbb{R}^n)$

collection of linear maps

Remark
$n = m$ Then $T_a^{-1} T_a = \text{Identity map}$.

In full generality

$$\boxed{T_b \, T_a = T_{ba}}$$

Note Many Books do not distinguish between the matrix a, and the linear map T_a.

Change of basis

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$T_a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ y \end{pmatrix}$$

v, w Spanning

$$z = \lambda v + \mu w \qquad \lambda, \mu \text{ suitable.}$$

$$T_a(z) = \tilde{\lambda} v + \tilde{\mu} w$$

New map from $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{\lambda} \\ \tilde{\mu} \end{pmatrix}$

Remark Every linear map is given by a matrix

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$T_a(v) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \hat\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \hat\mu \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$T_a(w) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -w$

$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \hat\lambda + \hat\mu \\ \hat\lambda - \hat\mu \end{pmatrix}$

$\hat\lambda = 2 \quad \hat\mu = 1$

$T_a(v) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cancel{\hat\mu} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The new matrix is given

$\tilde a = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$

correct now!

Theorem $\tilde a = \boxed{C^{-1} a C}$

C matrix of the basis change.

$C = (v\ w) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$