# LINEAR INDEPENCE
## and
# HIGHER DIMENSION

A set $\{v_1, \dots, v_k\}$ in $\mathbb{R}^n$ (or some vector space)
is called linearly independent if

$$\sum \alpha_j v_j = 0 \quad \text{implies} \quad \alpha_j = 0 \text{ for } j = 1 - , k.$$

A set $\{v_1 \dots v_k\} \subseteq \mathbb{R}^n$ is called basis if for every
$x \in \mathbb{R}^n$ there is a unique way of writing

$$x = \sum_{j=1}^{k} \alpha_j v_j \quad \text{with } \alpha_j \in \mathbb{R}$$

Pb Given $v_1 - v_k \in \mathbb{R}^n$ how can we check
that $v_1, \dots, v_k$ are linearly independent?

Motivation: $(k=n)$ for calculus
Let $a\ \mathbb{R}^k \to \mathbb{R}^n$ be the ~~linear~~ matrix such that

$$\boxed{a \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{j-th post} = v_j}$$

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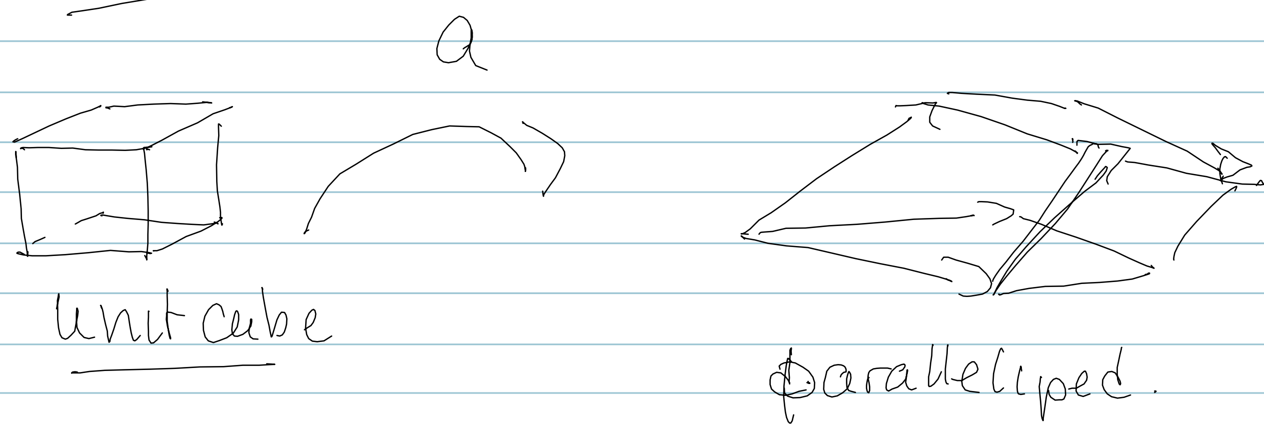
Then

$$\boxed{\mathrm{vol}_k \{ \textstyle\sum \alpha_j v_j : 0 \le \alpha_j \le 1 \} = |\det a|}$$




Solution 1 (Pb)

Consider the matrix $b = (v_i \cdot v_j)_{1 \le i,j \le k}$.

then $v_1 \dots v_k$ are linearly independent if and only if

$$\boxed{\det(b) > 0}$$

Recall the inductively defined determinant formula

$$\det a = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A^{1j})$$

where $A^{ij}$ is the matrix obtained by erasing the i-th row and j-th column.

Ex

$$\det\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = 1\cdot(\overbrace{45-48}^{-3}) - 2(\overbrace{36-42}^{-6}) + 3(\overbrace{32-35}^{-3})$$

$$= 1(-3) - 2(-6) + 3(-3)$$

$$= -3 + 12 - 9 = 0$$

Solution 2 GRAM-SCHMIDT-METHOD

Given $V_1 \ldots V_k$

Output a) $W_1 \ldots W_k$

b) $n_1, \ldots, n_k$.

Step 1
$W_1 = V_1$

Step 2 $W_2 = V_2 - P_{W_1}(V_2)$ $W_2 = 0$ Stop

⋮

Step k $W_k = V_k - P_{W_1}(V_k) - \ldots - P_{W_{k-1}}(V_k)$

$W_k \equiv 0$ Stop

If $W_k \neq 0$ and program did not stop, then $V_1 \ldots V_k$ are linearly independent. Then we can produce

$n_j = \frac{W_j}{|W_j|}$ which have

the property $|n_j| = 1$ $n_j \cdot n_\ell = 0$ $j \neq \ell$

Properties

a) If $w_j$ are never $0$, then $v_1 \dots v_k$ are lin. indep

b) $$w_j = v_j + \sum_{i=1}^{j-1} \alpha_{ij} v_i$$

c) $$v_j = w_j + \sum_{i=1}^{j-1} \beta_{ij} w_i$$

d) let $a$ be the matrix such that $a(e_j) = v_j$
~~and~~ and $b$ be the matrix such that $b(e_j) = w_j$.

Then there exists an upper triangular matrix $\alpha$

$$\alpha_{ij} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$ such that

$$\boxed{b = \alpha\, a}$$

( In particular $\det(b) = \det(a)$ $\left(\text{because } \det(\alpha) = 1\right)$

e) The matrix $b$ is easy to invert. Indeed

$x = \sum \lambda_j w_j$, then $$\boxed{\lambda_j = \frac{x \cdot w_j}{w_j \cdot w_j}}$$

gives the unique coefficients.

f) $\{\sum \lambda_j v_j \,;\, \lambda_j \in \mathbb{R}\} = \{\sum \lambda_j w_j \,;\, \lambda_j \in \mathbb{R}\}$ ↙ Same Span

Problem: let S be given by

$$x^2+y^2-z^2-w^2=5$$

Find an orthonormal basis for the tangent space

Solution: We understand S as a surface of the function

$$w=f\begin{pmatrix}x\\y\\z\end{pmatrix}=\sqrt{x^2+y^2-z^2-5}$$

Then the tangent space is given by the span of vectors

$$\begin{pmatrix}1\\0\\0\\\frac{\partial f}{\partial x}\end{pmatrix}\quad\begin{pmatrix}0\\1\\0\\\frac{\partial f}{\partial y}\end{pmatrix}\quad\begin{pmatrix}0\\0\\1\\\frac{\partial f}{\partial z}\end{pmatrix}$$

$$=\begin{pmatrix}1\\0\\0\\\frac{x}{\sqrt{\phantom{x}}}\end{pmatrix}\quad\begin{pmatrix}0\\1\\0\\\frac{y}{\sqrt{\phantom{x}}}\end{pmatrix}\quad\begin{pmatrix}0\\0\\1\\\frac{-z}{\sqrt{\phantom{x}}}\end{pmatrix}$$

Let $\gamma = (\sqrt{\quad})^{-1}$ we apply GS Method

$$W_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ x\gamma \end{pmatrix}$$

$$W_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ y\gamma \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ 0 \\ x\gamma \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ y\gamma \end{pmatrix}}{1+y^2\gamma^2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ x\gamma \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ y\gamma \end{pmatrix} - \frac{xy\gamma^2}{1+y^2\gamma^2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ x\gamma \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{xy\gamma^2}{1+y^2\gamma^2} \\ 1 \\ 0 \\ \frac{\gamma y + \gamma^3 y^3 - x^2 y \gamma^3}{1+y^2\gamma^2} \end{pmatrix} \qquad \frac{\gamma y(1+y^2\gamma^2) - x^2 y \gamma^3}{1+y^2\gamma^2}$$

$$\hat{W_2} = \begin{pmatrix} -\frac{xy\gamma^2}{1+y^2\gamma^2} \\ 0 \\ \gamma y + \gamma^3 y^3 - x^2 y \gamma^3 \end{pmatrix}$$

$$W_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -z\gamma \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \\ -\gamma z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ y\gamma \end{pmatrix}}{1+y^2\gamma^2} - \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \\ -\gamma z \end{pmatrix} \begin{pmatrix} -\frac{xy\gamma^2}{1+y^2\gamma^2} \\ 0 \\ \gamma y + \gamma^3 y^3 - y^2 y \gamma^3 \end{pmatrix}}{-\|} \begin{pmatrix} \quad \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\gamma z \end{pmatrix} + \frac{\gamma^2 yz}{1+y^2\gamma^2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ y\gamma \end{pmatrix} + \frac{\gamma z(\gamma y+\gamma^3 y^3-\gamma^3 x^2 y)}{(\gamma y+\gamma^3 y^3-\gamma^3 x^2 y)^2} \begin{pmatrix} -xy\gamma^2 \\ 1+y^2\gamma^2 \\ 0 \\ - \end{pmatrix}$$

Okay I agree not pretty, but its clear that that can be calculated

For example for $x=2, y=5, z=4, w=2\sqrt{2}$

Here $\gamma = \frac{1}{\sqrt{5}}$ ...

Or for $x = \cancel{\sqrt{5}}$ $y = z = \cancel{\ldots}\ 0$ $w = \sqrt{11}$

we find $\gamma = \frac{1}{\sqrt{4}} = \frac{1}{2}$

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$h_1 = \begin{pmatrix} 1/\sqrt{5} \\ 0 \\ 0 \\ 2/\sqrt{5} \end{pmatrix} \quad h_2 = w_2 \quad h_3 = w_3 \quad \square$$

Note For some problems it suffices to know the normal vector to the hypersurface.

Here we can use the cross product

$$\det\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ v_1 \longrightarrow & & & \\ v_2 \longrightarrow & & & \\ v_3 \longrightarrow & & & \end{pmatrix} = n$$

In our example

$$\begin{pmatrix} e_1 & 1 & 0 & 0 \\ e_2 & 0 & 1 & 0 \\ e_3 & 0 & 0 & 1 \\ e_4 & 2 & 0 & 0 \end{pmatrix}$$

$$= e_1 \det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} - e_2 \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

$$+ e_3 \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} - e_4 \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2e_1 - 0e_2 + 0e_3 - e_4 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Than it is easy to find ~~normal~~ an ONB

$$\cancel{x_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \end{pmatrix} \frac{1}{\sqrt{3}}} \qquad x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \frac{1}{\sqrt{3}}$$

$$x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

which also span the tangent space ?!