Fubini and Cavalieri

Definition 0.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$, we say that f has bounded support if there exists $R > 0$ such that*

$$f(x) = 0$$

for all $|x| \geq R$.

Theorem 0.2 *(Fubini) Let $1 \leq k \leq n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function (with bounded support). Then*

$$\int_{\mathbb{R}^n} F(x_1, ..., x_n) dV_n(x_1, ..., x_n)$$
$$= \int_{\mathbb{R}^k} \big(\int_{\mathbb{R}^{n-k}} F(x_1, ..., x_k, x_{k+1}, ..., x_n) dV_{n-k}(x_{k+1}, ..., x_n) \big) dV_k(x_1, ..., x_k) .$$

Moreover, the integrals inside makes sense.

Theorem 0.3 *(Cavalieri, Fubini-Tonnelli). Let $1 \leq k \leq n$ and $F : \mathbb{R}^n \to f$ be a positive function such that*

1. *For fixed (frozen) $(x_1, ..., x_k)$ the function which sends $(x_{k+1}, ..., x_n)$ to the value $F(x_1, ..., x_k, x_{k+1}, ..., x_n)$ is integrable;*
2. *For fixed (frozen) $(x_1, ..., x_k)$ the function which sends $(x_{k+1}, ..., x_n)$ to*

$$\int_{\mathbb{R}^{n-k}} F(x_1, ..., x_k, x_{k+1}, ..., x_n) dV_{n-k}(x_{k+1}, ..., x_n)$$

   is integrable.

Then F is integrable and

$$\int F(x_1, ..., x_n) dV_n(x_1, ..., x_n)$$
$$= \int_{\mathbb{R}^n} \big(\int_{\mathbb{R}^k} F(x_1, ..., x_k, x_{k+1}, ..., x_n) dV_{n-k}(x_{k+1}, ..., x_n) \big) dV_k(x_1, ..., x_k) .$$

Remark 0.4 *You will usually apply this by checking that the "frozen" functions are continuous (or have only only finitely many discontinuities) in 1. and 2.*