Fubini and Cavalieri

**Definition 0.1** Let \( f : \mathbb{R}^n \to \mathbb{R} \), we say that \( f \) has bounded support if there exists \( R > 0 \) such that
\[
f(x) = 0
\]
for all \( |x| \geq R \).

**Theorem 0.2** (Fubini) Let \( 1 \leq k \leq n \) and \( f : \mathbb{R}^n \to \mathbb{R} \) be an integrable function (with bounded support). Then
\[
\int_{\mathbb{R}^n} F(x_1, \ldots, x_n) dV_n(x_1, \ldots, x_n)
= \int_{\mathbb{R}^k} \left( \int_{\mathbb{R}^{n-k}} F(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) dV_{n-k}(x_{k+1}, \ldots, x_n) \right) dV_k(x_1, \ldots, x_k).
\]
Moreover, the integrals inside makes sense.

**Theorem 0.3** (Cavalieri, Fubini-Tonnelli). Let \( 1 \leq k \leq n \) and \( F : \mathbb{R}^n \to f \) be a positive function such that

1. For fixed (frozen) \((x_1, \ldots, x_k)\) the function which sends \((x_{k+1}, \ldots, x_n)\) to the value \( F(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \) is integrable;

2. For fixed (frozen) \((x_1, \ldots, x_k)\) the function which sends \((x_{k+1}, \ldots, x_n)\) to
\[
\int_{\mathbb{R}^{n-k}} F(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) dV_{n-k}(x_{k+1}, \ldots, x_n)
\]
is integrable.

Then \( F \) is integrable and
\[
\int F(x_1, \ldots, x_n) dV_n(x_1, \ldots, x_n)
= \int_{\mathbb{R}^k} \left( \int_{\mathbb{R}^{n-k}} F(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) dV_{n-k}(x_{k+1}, \ldots, x_n) \right) dV_k(x_1, \ldots, x_k).
\]

**Remark 0.4** You will usually apply this by checking that the “frozen” functions are continuous (or have only only finitely many discontinuities) in 1. and 2.