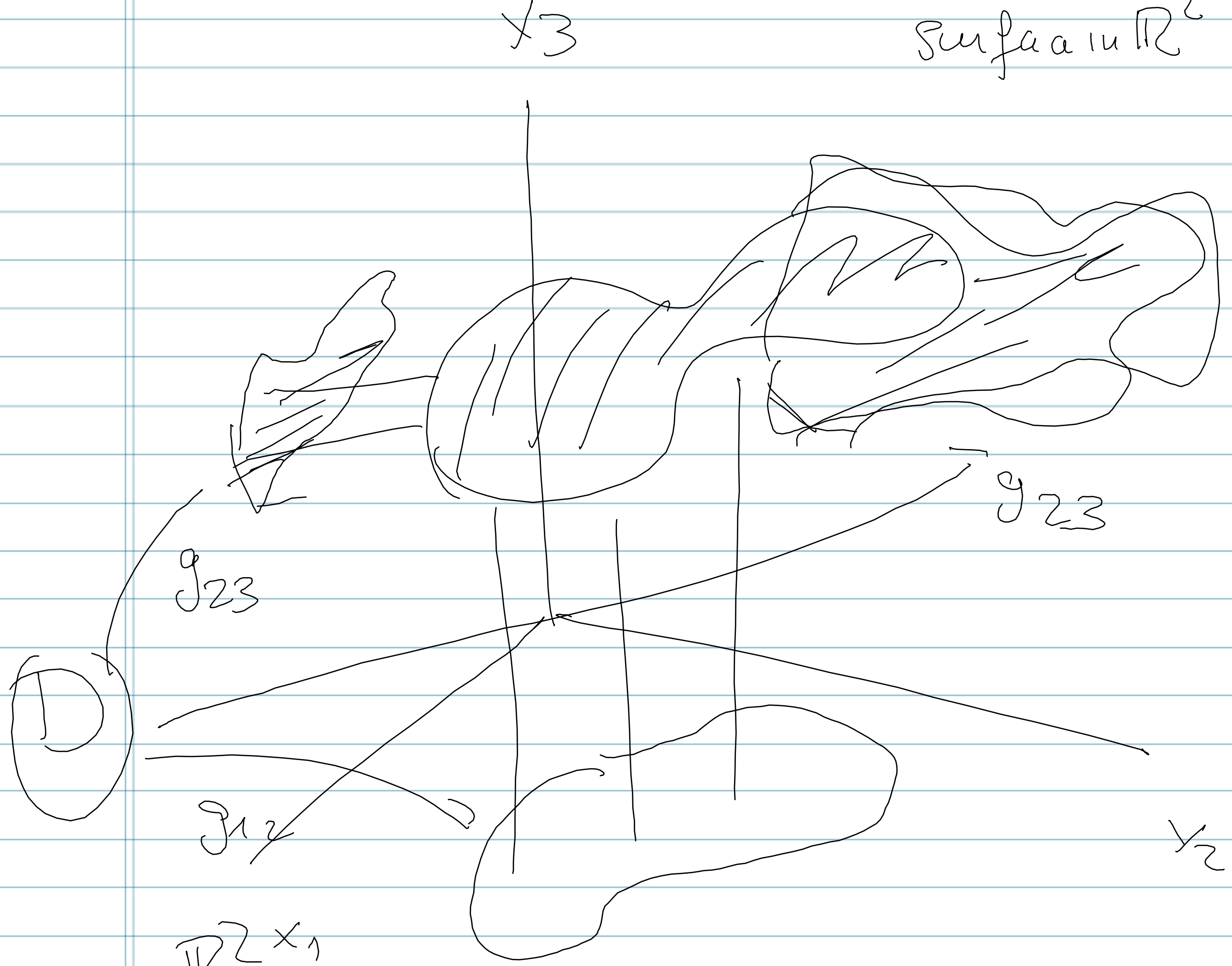


$\mathbb{R}^2$

$\cup$

$g \quad D \longrightarrow \mathbb{R}^3$

g

Divergence theorem:

Recall $w = P\, v_2 v_3 + Q\, v_3 v_1 + R\, v_1 v_2$

① $$dw = \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_2}\right) v_1 v_2 v_3$$

General Stokes's $B \subset \mathbb{R}^3$ oriented ∂B border

$$\int_B dw = \int_{\partial B} w$$

$$\begin{tikzcd}
\int_B dw = \int_{\partial B} w \arrow[d, phantom, "\parallel"] \arrow[dr, phantom, "\parallel"] & \\
\int_B dw(V)\, d(x_1 x_2 x_3) & \int_{\partial B} V \cdot n \, dS
\end{tikzcd}$$

Here: $$\boxed{dw\begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3}}$$

lemma $V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$

$$V \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= V_1 (a_2 b_3 - a_3 b_2) + V_2 (a_3 b_1 - a_1 b_3) + v_3 (a_1 b_2 - a_2 b_1)$$

$$= V_1 \det \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} + V_2 \det \begin{pmatrix} a_1 & b_1 \\ a_3 & b_3 \end{pmatrix} + v_3 \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

Particular case: $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\frac{\partial g}{\partial u} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \frac{\partial g}{\partial w} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\boxed{V \cdot \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial w} = V_1 \det g'_{23} + V_2 \det g'_{31} + V_3 \det g'_{12}}$$

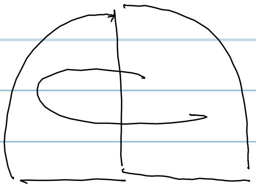
Prop

$$\boxed{\int_{\partial S} P\, v_2 v_3 + Q\, v_3 v_1 + R\, v_1 v_2 = \int_S \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot n \; dS}$$

(first)

Examples for divergence thm

$f(x) = (1-x^p)^{1/q}$



$$B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{array}{l} x^2+y^2 \le 1 \\ 0 \le z \le \left(1-(x^2+y^2)^{p/2}\right)^{1/q} \end{array} \right\}$$

$V = \begin{pmatrix} 0 \\ 0 \\ \frac{z^2}{2} \end{pmatrix}$ vector field

$$\boxed{\int_{\partial B} V \cdot n \; dS = \text{average gravitational force on boundary}}$$

Thus

$$\int_B z\, d(x,y,z) = \int_{\partial B} \frac{z^2}{2} \overbrace{dx \wedge dy}^{V_1 \cdot V_2}$$

Proof $w = \frac{z^2}{2} V_1 V_2$

$$\boxed{\begin{aligned} dw &= z\, V_1 V_2 V_3 \\ &= z\, d(x_1, x_2, x_3) \end{aligned}}$$

Both calculation

$$\int_B z\, d(x,y,z) = \int_{x^2+y^2 \le 1} \int_0^{(1-r^p)^{1/q}} z\, dz \quad dA(x,y)$$

$$= \frac{2\pi}{2} \int_0^1 (1-r^p)^{2/q} \quad r\, dr$$

$$= \pi \int_0^1 (1-r^p)^{2/q}\, r\, dr$$

(α) $P = 0$

$Q = 2x_1 \quad R = x_2 \quad = 3\pi$

$2\int_0^{2\pi} \cos^2(\theta)\, d\theta + \int_0^{2\pi} \sin^2(\theta)\, d\theta$

$- \int^1 f'(r)\, r\, dr$

$= 3 \int_0^1 f(r)\, dr \quad \pi$

$\boxed{\text{Integration by parts!}}$

The other

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ (1-(x^2+y^2)^{p/2})^{q} \end{pmatrix}$$

$$g'_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\int_D (1 - \underline{r^p})^{2/q}_{2}\, dA(x,y) = \pi \int_0^1 (1-r^p)^{2/q}\, r\, dr.$$

Remark Integral can be calculated explicitely

for $p = 1$ (Integration by parts)

$p = 2$ ($u = 1 - r^2$ substitution)

Application: $f(1) = 0 \quad B_f =$  rotation

$w = P\, dx_1 \wedge dx_2 + Q\, dx_2 \wedge dx_3 + R\, dx_3 \wedge dx_1$

$dw = \left(\frac{\partial P}{\partial x_3} + \frac{\partial Q}{\partial x_1} + \frac{\partial R}{\partial x_2} \right)$

$$\boxed{\begin{aligned} \int_{B_f} dw &= \int_D P(g(s,t))\, dA(s,t) + \int Q(g(s,t))\,(-f')\,\frac{s}{\sqrt{s^2+t^2}}\, dA(s,t) + \int R(g)(-f')\left(\frac{t}{\sqrt{s^2+t^2}}\right) dA \\ &\ne \int_D P\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) dA(s,t) \end{aligned}}$$

$\longrightarrow$ (α)

Stokes thm let $S \subset \mathbb{R}^3$ be an oriented surface with boundary $\gamma[a,b] = \partial S$

$$\int_{\partial S} P\,dx_1 + Q\,dx_2 + R\,dx_3$$

$$= \int_S \begin{pmatrix} \frac{\partial R}{\partial x_2} - \frac{\partial Q}{\partial x_3} \\ \frac{\partial Q}{\partial x_3} - \frac{\partial R}{\partial x_1} \\ \frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \end{pmatrix} \cdot n \; dS$$

Proof $\omega = P\,dx_1 + Q\,dx_2 + R\,dx_3$

$$d\omega = \frac{\partial P}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial P}{\partial x_3} dx_3 \wedge dx_1$$

$$+ \frac{\partial Q}{\partial x_1} dx_1 \wedge dx_2 + \frac{\partial Q}{\partial x_3} dx_3 \wedge dx_2$$

$$+ \frac{\partial R}{\partial x_1} dx_1 \wedge dx_3 + \frac{\partial R}{\partial x_2} dx_2 \wedge dx_3$$

$$= \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial Q}{\partial x_3} - \frac{\partial R}{\partial x_1}\right) dx_3 \wedge dx_1$$

$$+ \left(\frac{\partial R}{\partial x_2} - \frac{\partial Q}{\partial x_3}\right) dx_2 \wedge dx_3$$

In $\mathbb{R}^3$ everything can be formulate with vector fields

Short version

$$\boxed{\int_{\partial S} V \cdot T \, ds = \int_S \overbrace{(\nabla \times V)}^{\mathrm{curl}(V)} \cdot n \, dS}$$

$$\nabla \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$$

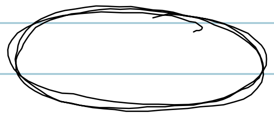
$$= \begin{pmatrix} \frac{\partial}{\partial x_2} Q - \frac{\partial}{\partial x_3} R \\ \frac{\partial}{\partial x_3} P - \frac{\partial}{\partial x_1} R \\ \frac{\partial}{\partial x_1} Q - \frac{\partial}{\partial x_2} P \end{pmatrix}$$

Example $\nabla \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{z^2}{2} \end{pmatrix}$

$\frac{\partial}{\partial x_2} Q = 0 \quad R, R = 0$

$$\begin{pmatrix} 0 \\ \frac{\cancel{x_1} x_3^2}{2} \\ 0 \end{pmatrix}$$

$\partial B = S^{up} \cup S^{down}$

$\partial S =$ 

$$\int_{\partial B} N \cdot n \, dS = \pi \int_0^1 (1-r^p)^{2/q} \, dr$$

$$\boxed{V\left(\begin{matrix}\cancel{x_1}\\ \cancel{x_2}\\ y\end{matrix}\right) = 0}$$

$\|$

$$\int_{S^{up}} N \cdot n \, dS$$

(1

$$= \int_{\partial S^{up}} \nabla \times V \, dS$$

$$\nabla \cdot V = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \alpha \begin{pmatrix} 0 \\ \frac{x_1 x_3^2}{2} \\ J \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{0} - x_1 x_3 \\ 0 \\ + \frac{x_3^2}{2} \end{pmatrix}$$