Distance from a subspace

Let \( V \) be an inner product space

\[ |v| = (v|v|^2) \text{ length of a vector} \]

Distance \( p, q \in V \) \( \text{dist}(p, q) = |p - q| \)

Let \( p \in V \) \( W \subseteq V \) be a subspace (of finite dimension)

Find \( w \in W \) such that

1) \( \sum \langle w, e \rangle e = |w - p| \)

2) \( |w - p| \)

Solution: Choose an orthonormal basis \( b_1, \ldots, b_k \) of \( W \)

Define

\[ w = \sum \langle w, b_i \rangle b_i \]

Define \( z = p - w \)

Claim 1) \( (z|b_j) = 0 \) for all \( j \)

Indeed \( \langle z, p - w \rangle = \langle z, b_1 \rangle = \sum \langle z, b_i \rangle b_i \)

= \( \sum (y_i | p) - (y_i | b_i) (b_i | p) \)

= \( \sum (y_i | p) - (y_i | b_i) \sum \langle b_i, b_j \rangle = 0 \)

Claim 2) \( (z|w) = 0 \) for all \( w \in W \)

Indeed \( (z|w) = z| \sum \langle w, b_i \rangle b_i = \sum \langle z, b_i \rangle (b_i | b_i) = 0 \)
Claim: \[ |p-w|^2 \geq |p-\bar{w}|^2 \quad \text{for all } \bar{w} \]
and equality only holds for \( \bar{w} = w \).

Indeed,

\[
|p-w|^2 = |p-\bar{w} + (w-\bar{w})|^2 = (p-\bar{w})(p-\bar{w}) + (w-\bar{w})(w-\bar{w}) + (\bar{w} - w)(\bar{w} - w) = 0
\]

\[
|p-w|^2 + |w-\bar{w}|^2
\]


Thus, \( w = \sum (x_i \mathbf{v}_i) \mathbf{v}_i \) is the solution and \( |p-w| \) is the minimal distance.

**Def:** \( W \subseteq V \) we call

\[
\text{Proj}_W(p) = \sum (x_i \mathbf{v}_i) \mathbf{v}_i
\]

the projection of \( p \) onto \( W \). This projection does not depend on the choice of the orthonormal basis.

\[
\mathbf{v} \parallel \mathbf{w} \quad \text{and} \quad \mathbf{z} \perp \mathbf{w}
\]

**Def:** \( \cos(p, W) = \frac{(\mathbf{v} \cdot \mathbf{p}) V}{|V||V|} = \text{Proj}_W(p) \)

where \( W = \text{ker} t \in R \) is a subspace.
Comment: let $V$ be an inner product space, $W \leq V$ a finite dimensional subspace, and $p \in V$ be a point.

By choosing an orthonormal basis

$$b_1, \ldots, b_k$$

for $W$ and the additional orthonormal vector

$$b_{k+1} = \frac{p - w}{||p - w||} \quad w = \sum_{k} (b_k \cdot p) b_k$$

we may transport the whole picture to

$$\mathbb{R}^{k+1}$$

$\mathbb{R}^k$ corresponds to $W$

via the map $(t_1, \ldots, t_k) \mapsto \sum_{k} b_k t_k$

$(0, \ldots, 0)$ corresponds to $b_{k+1}$

$$p = ||p - w|| b_{k+1} + \frac{w}{||w||} w$$

$$z = ||p - w|| b_{k+1} = p - w$$

$W$ subspace

Angle between $p$ and $w$:

$$\cos \theta = \frac{(b_k \cdot p)}{||b_k||}$$