Practice problems

(1) Formulate the root test and give an example.

(2) Give an example were it is easier to apply the ratio test than to apply the root test.

(3) Does the series converge? Justify your answer!
   (a) $\sum_{n} n^{\sqrt{\ln n}}$
   (b) $\sum_{n} n^{-\sqrt{\ln n}}$
   (c) $\sum_{n \text{ prime}} \frac{1}{n^2}$
   (d) $\sum_{n \text{ prime}} 2^{-n}$
   (e) $\sum_{n} 2^{-n^2}$
   (f) $\sum_{n} (-1)^n (\sqrt{n + 1} - \sqrt{n})$
   (g) $\sum_{n} (-1)^n ((n + 1)^{1/3} - n^{1/3})$
   (h) $\sum_{n} \frac{\sqrt{n} + 1}{n^3 + 2}$
   (i) $\sum_{n} a_n$. Here $a_n$ is defined recursively by $a_1 = 1$, $a_{n+1} = \frac{2 + \cos(n)}{\sqrt{n}}$.

(4) Apply alternating series to approximate the series up to 0.01. How many terms do you need? DON’T WRITE DOWN THE APPROXIMATION!
   $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$$
   $$\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln n}$$

(5) Apply the integral test and approximate to approximate the series up to 0.01. How many terms do you need? DON’T WRITE DOWN THE APPROXIMATION!
   $$\sum_{n} \frac{\sqrt{n} + 1}{n^3 + 2}$$
   $$\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$$
   $$\sum_{n} \frac{1}{n \sqrt{3} + n^2}$$
solutions

1): Let \((a_n)\) be a sequence of non-zero real numbers such that
\[
\lim_n |a_n|^{1/n} = L < 1.
\]
Then \(\sum_n a_n\) converges absolutely. If the limit exists and \(L > 1\), then \(\sum_n a_n\) diverges.
If the limit does not exist or \(L = 1\), no conclusion can be made.

2) For \(a_n = \frac{x^n}{n!}\), it is easier to apply the ratio test. Indeed,
\[
\lim_n \frac{|a_{n+1}|}{|a_n|} = \lim_n \frac{|x|}{n+1} = 0
\]
and hence the root test applies. It is harder to apply the root test
\[
\lim_n \left(\frac{|x|^n}{n!}\right)^{1/n} = \lim_n \frac{|x|}{(n!)^{1/n}} = 0.
\]
Indeed, we proved in class (using some funny integrals) that
\[
e^{n-1} \frac{n^n}{n!} \leq e^n.
\]
Taking the \(n\) root gives
\[
e \frac{1}{(en)^{1/n}} \leq \frac{n}{(n!)^{1/n}} \leq e.
\]
Note that the left hand side converges to \(e\) and hence
\[
\lim_n \frac{1}{n!^{1/n}} \leq \lim_n \frac{e}{n} = 0.
\]

3a) \(n^{\ln n}\) does not converge to 0. Indeed, for \(n \geq e^2\) we have \(n^{\ln n} \geq n\) which goes to infinity. By the simple version of the Cauchy test \(\sum_n n^{\ln n}\) is diverging.

3b) By the same argument we note that for \(n \geq e^4\) we have \(\sqrt{\ln n} \geq \sqrt{4} = 2\). By comparison with \(\frac{1}{n^2}\) (obtained from integral test), we deduce convergence.

3c) That is a particular case of the subsum’s test, because \(\sum_n \frac{1}{n^2}\) is absolutely convergent.

3d) As above with \(c_n = 2^{-n}\) and the subsum test.

In 3c) and 3d) we use \(I = \{p \mid p\text{ prime}\}\).

3e) Again subsum for \(2^{-n}\), we just take \(I = \{n\text{|n square}\}\).

3f) Here we want to use alternating series. We have to show that \(a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}\) converges to 0, which is obvious, is positive, obvious, and decreasing. But certainly
\[
\sqrt{n+1} + \sqrt{n+1} \geq \sqrt{n} + \sqrt{n+1}.
\]
Thus alternating series test applies.
3g) Same procedure as in 3f). We note that \( a_n = (n+1)^{1/3} - n^{1/3} \) is positive and

\[
a_{n+1} \leq a_n.
\]

We define \( f(x) = (x+1)^{1/3} - x^{1/3} \) and calculate

\[
f'(x) = \frac{1}{3} \left[ (x+1)^{-2/3} - x^{-2/3} \right] \leq 0.
\]

Thus \( f \) is decreasing and hence \( a_{n+1} \leq a_n \). It remains to show

\[
\lim_n a_n = 0.
\]

Indeed, we have

\[
a_n = n^{1/3} \left[ (1 + \frac{1}{n})^{1/3} - 1 \right].
\]

By the fundamental theorem, we have

\[
(1 + \frac{1}{n})^{1/3} - 1 = \frac{1}{3} \int_1^{1+1/n} x^{-2/3} dx \leq \frac{1}{3} (1 + 1/n - 1) = \frac{1}{3n}.
\]

Thus \( \lim_n n^{-2/3} = 0 \) implies the assertion.

h) We compare with \( n^{-3/2} \) which is absolutely convergent thanks to integral test.

i) By induction we prove

\[
|a_n| \leq \frac{3^{n-1}}{\sqrt{(n-1)!}} = c_n.
\]

The right hand equality is a definition. Indeed, for \( n = 1 \) with the interpretation \( 0! = 1 \) this is true. Then \( |a_n| \leq \frac{3^{n-1}}{\sqrt{(n-1)!}} \) implies \((|2 + \cos(n)| \leq 3)\)

\[
|a_{n+1}| \leq \frac{3}{\sqrt{n}} |a_n| \leq \frac{3^n}{\sqrt{n!}}.
\]

Now we apply ration test to \( c_n \) and get

\[
\frac{c_{n+1}}{c_n} = \frac{3^{n-1}(n-2)}{\sqrt{n}} = \frac{3}{\sqrt{n}}
\]

This obviously converges to 0 and hence, the comparison test yields convergence.

(4) Easy as \( \pi \): we just need

\[
\frac{1}{(2n)!} \leq 0.01
\]

or \( 100 \leq (2n)! \). Note the sequence for factorials \( 1, 2, 6, 24, 125 = 5! \). Thus \( n = 3 \) will work. For the second we need \( 100 \leq \ln n \). Since \( \ln 3 \geq 1 \), we can take \( n = 3^{100} \).
(5) Easy solution for the first and the last problem. We use

\[ \frac{\sqrt{n} + 1}{n^3 + 2} \leq 2n^{-5/2} \]

and

\[ \frac{1}{n\sqrt{3 + n^2}} \leq n^{-2}. \]

Then we note that

\[ \left| \sum_{k=n+1}^{\infty} a_k \right| \leq \int_{n}^{\infty} f(x) \]

holds in both cases, where \( f(x) = 2x^{-5/2} \) in the first place and \( f(x) = x^{-2} \) in the second case. Thus we need

\[ 2 \int_{n}^{\infty} x^{-5/2} \, dx = \frac{4}{3} n^{-3/2} < 0.01 \]

or \( 400 \leq 3n^{3/2} \) or \( 160000 \leq 3n^3 \). \( n = 100 \) definitely works, but one can do better.

For the second case we need \( n^{-1} < 0.01 \) or \( n = 101 \).

For the middle problem we just estimate \( \ln n \geq 1 \) for \( n \geq 3 \) and compare with \( \frac{1}{n^2} \) where it is easy to integrate

\[ \int_{n}^{\infty} \frac{dx}{x^2} = \frac{1}{n}. \]

Then

\[ \frac{1}{n} \leq 0.01 \]

means \( n = 101 \) works.