Some words on analytic functions

Here we want to study the question whether

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

holds for $|x-x_0| < R$. Let us say that $f$ is represented by the McLaurin series at $x_0$ if we have the equality above and the power series on the right has a positive radius of convergence. We try to find some conditions which tell us that Taylor’s theorem

$$|f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)| \leq \frac{M^{(n+1)}(x)}{(n+1)!}|x-x_0|^{n+1}$$

can actually be applied because the bound

$$M^{(n+1)}(x) \geq |f^{(n+1)}(y)|$$

for all $|y-x_0| \leq |x-x_0|$ does not explode for large $n$.

We say a function $f$ defined on an open subset of the complex plain is complex differentiable at $z$ if

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists.

Remark 1. The same rules for differentiation you know form real functions are valid here:

1. Then sum, and produce of differentiable functions are differentiable. The same is true for the quotient, as long as there is no 0 in the denominator.
2. The composition of differentiable functions is differentiable.
3. Inverse functions of differentiable as far as they are defined.

Example 2. (1) Polynomials, rational functions away from their singularity.
(2) $f(x) = e^x$, $f(x) = \cosh(x)$, $f(x) = \sinh(x)$.
(3) $f(x) = \ln x$ is analytic on $\mathbb{C} \setminus \{x+i0 : x \leq 0\}$.

Theorem 3. Let $f$ be continuous around $x_0$. The following are equivalent

i) There exists a complex differentiable function $F$ in some ball $B_R(x_0) = \{z : |z-x_0| < R\}$ such that $F(x) = f(x)$ for $|x-x_0| < R$.
ii) $f$ is represented by the McLaurin series at $x_0$. 

We can only prove ii) implies i). Indeed, we may simply define
\[ F(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(z - x_0)^k. \]
Everything we said about how to differentiate a power series also applies for complex \(|z - x_0| < R\). More precisely for \(r \leq R\) the partial sums are uniformly convergent. The converse and harder implication i) implies ii) also shows the that the radius of convergence is the largest radius \(R\) such that we can find a complex differentiable function \(F\) on \(B_R(x_0)\) which coincides with \(f\) for real values.

**Definition 4.** We say that \(f\) is analytic around \(x_0\) if one of the conditions above hold. Analytic is just a shortcut for \(f\) is represented by the McLaurin series at \(x_0\).

**Example 5.** \(f(x) = \frac{1}{1+x^2}\). The complex 0 are ±\(i\). Thus for \(x_0 = 0\) we find that \(R = 1\), because the rational function is differentiable as long as we don’t hit the roots of the denominator.

**Example 6.** \(f(x) = x^\alpha\) for \(x_0 > 0\). We write
\[ f(x) = e^{\alpha \ln(x)}. \]
Then we see that \(f\) is analytic on \((0, \infty)\), thanks to the ln. Thus the radius of convergence at \(x_0\) is \(x_0\). Alternatively, we may use Newton’s binomial theorem
\[ (1 + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} y^k. \]
Now, we perform a change of variable
\[ x = K(1 + y) \]
with \(x_0\) corresponding to 0, i.e. \(K = x_0\). This gives \(K = x_0\), \(y = \frac{(x - x_0)}{x_0}\) and hence
\[ x^\alpha = x_0^\alpha (1 + y)^\alpha = x_0^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} y^k = x_0^\alpha \sum_{k=0}^{\infty} x_0^{-k} \binom{\alpha}{k} (x - x_0)^k. \]
Since \(\lim_{k \to \infty} |\binom{\alpha}{k}|^{1/k} = 1\) we find \(R = x_0\) as expected.