

# Holomorphic maps from the complex unit ball to Type IV classical domains

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**Abstract**

## 1 Introduction

The first part of this paper is devoted to establish new rigidity results for proper holomorphic maps from the complex unit ball to higher rank bounded symmetric domains. The rigidity properties have been extensively studied in the past decades for proper holomorphic maps  $F : \Omega_1 \rightarrow \Omega_2$ , between bounded symmetric domains  $\Omega_1, \Omega_2$ . The pioneer works are due to Poincaré [P] and later to Alexander [Al] when  $\Omega_1, \Omega_2$  are complex unit balls. In particular, any proper holomorphic self-map of the unit ball in  $\mathbb{C}^n$  is an automorphism if  $n \geq 2$  [Al]. It is well-known that the rigidity properties fail dramatically for proper holomorphic maps between balls of different dimensions. For this type of results, see [HS], [L], [Fo1],[G1], [St], [Do], [D1] and etc. However, the rigidity properties can still be expected if certain boundary regularity of the map is assumed. See [W], [Fa], [CS], [Hu1], [Hu2], [HJ], [HJY], [Eb] and etc. The lists above are by no means to be complete.

On the other hand, it is a widely open problem to understand proper holomorphic maps  $F : \Omega_1 \rightarrow \Omega_2$  between bounded symmetric domains  $\Omega_1, \Omega_2$  of higher rank. When  $\text{rank}(\Omega_1) \geq \text{rank}(\Omega_2) \geq 2$  and  $\Omega_1$  is irreducible, it was proved by Tsai [Ts] that  $F$  must be a totally geodesic isometric embedding with respect to Bergman metrics. Tu [Tu1] proved that the proper holomorphic map between equal dimensional irreducible bounded symmetric domains must be an automorphism. When  $\text{rank}(\Omega_2) \geq \text{rank}(\Omega_1)$ , the studies are mainly focused on the Type I classical domains and many interesting results have been established (cf. [Tu2], [Ng4], [KZ1], [KZ2] et al). Note that the total geodesy of  $F$  fails in general although it is believed that  $F$  should take certain special forms module automorphisms (cf. [Ng4] [KZ2]). In this paper, we prove new rigidity results for proper holomorphic maps from the unit ball in  $\mathbb{C}^n$  to

the  $m$ -dimensional Type IV classical domain. The isometry property still survives although the total geodesy fails in our setting. In particular, we establish a classification result for CR maps between their boundaries when  $m = n + 1$ .

Write  $\mathbb{B}^n$  for the unit ball in  $\mathbb{C}^n$ , and  $D_m^{IV}$  for the classical type IV domain in  $\mathbb{C}^m$  and equip them with the Bergman metrics  $\omega_{\mathbb{B}^n}, \omega_{D_m^{IV}}$ , respectively (See section 2 for explicit formulas). We say a holomorphic map  $F : \mathbb{B}^n \rightarrow D_m^{IV}$  is an isometric embedding or simply an isometry if  $F^*(\omega_{D_m^{IV}}) = \lambda\omega_{\mathbb{B}^n}$  for some constant  $\lambda > 0$ . Write  $\partial\mathbb{B}^n$  and  $\partial D_m^{IV}$  for the unit sphere in  $\mathbb{C}^n$  and the boundary of  $D_m^{IV}$ , respectively.

**Theorem 1.1.** *Assume  $n \geq 4, n + 1 \leq m \leq 2n - 3$ .*

1. *(Local version) Let  $F$  be a holomorphic map from a connected open set  $U$  in  $\mathbb{C}^n$  containing  $p \in \partial\mathbb{B}^n$  to  $\mathbb{C}^m$ . Assume that  $F(\partial\mathbb{B}^n \cap U) \subset \partial D_m^{IV}$  and  $F(U) \not\subset \partial D_m^{IV}$ . Then  $F$  extends to a holomorphic isometric embedding from  $\mathbb{B}^n$  into  $D_m^{IV}$  with  $F^*(\omega_{D_m^{IV}}) = \frac{m}{n+1}\omega_{\mathbb{B}^n}$ .*
2. *(Global version) Any algebraic proper holomorphic map from  $\mathbb{B}^n$  to  $D_m^{IV}$  is an isometric embedding with  $F^*(\omega_{D_m^{IV}}) = \frac{m}{n+1}\omega_{\mathbb{B}^n}$ .*

Recall that a holomorphic function  $f$  over  $U \subset \mathbb{C}^n$  is called (holomorphic) Nash-algebraic, or simply algebraic if there is an irreducible holomorphic polynomial  $P(z, X)$  in  $X$  with coefficients polynomials of  $z$  such that  $P(z, f(z)) \equiv 0$  over  $U$ . A holomorphic map  $F$  is called algebraic if each of its components is algebraic. In the case  $m = n + 1$ , the proper holomorphic maps are classified under a weaker boundary regularity condition.

**Theorem 1.2.** *Let  $n \geq 4$  and  $F$  be a  $C^2$ -smooth CR transversal map from an open piece of  $\partial\mathbb{B}^n$  to an open smooth piece of  $\partial D_{n+1}^{IV}$ . Then  $F$  extends to a holomorphic isometry from  $\mathbb{B}^n$  to  $D_{n+1}^{IV}$ . Furthermore,  $F$  is equivalent to either*

$$\left( z_1, \dots, z_{n-1}, \frac{\frac{1}{2} \sum_{i=1}^{n-1} z_i^2 - z_n^2 + z_n}{\sqrt{2}(1 - z_n)}, \sqrt{-1} \frac{\frac{1}{2} \sum_{i=1}^{n-1} z_i^2 + z_n^2 - z_n}{\sqrt{2}(1 - z_n)} \right); \quad (1)$$

or

$$\left( z_1, \dots, z_{n-1}, z_n, 1 - \sqrt{1 - \sum_{j=1}^n z_j^2} \right). \quad (2)$$

Here two proper holomorphic maps  $F_1, F_2 : \Omega_1 \rightarrow \Omega_2$  are called equivalent if there exist automorphisms  $\phi$  and  $\psi$  of  $\Omega_1$  and  $\Omega_2$  respectively, such that  $\psi \circ F_1 \circ \phi = F_2$ . The definition of CR transversality will be given in Section 2. We next list some important remarks of Theorem 1.1 and Theorem 1.2.

**Remark 1.3.** 1. *It follows from Mok's result ([M4]) that the algebraicity assumption on  $F$  is necessary in Theorem 1.1. Moreover, a non-algebraic proper map from  $\mathbb{B}^n$  to  $D_{n+2}^{IV}$  that is not an isometry will be constructed in Example 2.4.*

2. *The statement of Theorem 1.2 fails if the transversality assumption is dropped. Similarly, the statement of Theorem 1.1 fails if the condition  $F(U) \not\subset \partial D_m^{IV}$  is dropped. See Example 2.5.*
3. *The conclusion of Theorem 1.1 fails if  $m \geq 2n$  (cf. Example 2.6). The conclusion of Theorem 1.2 fails if  $n = 1$  (cf. Example 2.7). We suspect that Theorem 1.1 holds for all  $n \geq 2, n + 1 \leq m \leq 2n - 1$  and Theorem 1.2 holds for all  $n \geq 2$ . See more details in Section 2.*
4. *We state the following fact as a remark of Theorem 1.2. Let  $G$  be a nonconstant local  $C^2$ -smooth CR map from the unit sphere in  $\mathbb{C}^n$  to  $\partial D_{n+1}^{IV}$ . Then the image of  $G$  cannot be contained in the singular set of  $\partial D_{n+1}^{IV}$ . This is due to the fact that the singular set of  $\partial D_{n+1}^{IV}$  is a real  $(n + 1)$ -dimensional subvariety of  $\{Z\bar{Z}^t = 2\}$ .*

The study of proper holomorphic maps from  $\mathbb{B}^n$  to  $D_m^{IV}$  or CR maps from  $\partial\mathbb{B}^n$  to  $\partial D_m^{IV}$  is closely related to CR maps between hyperquadrics. Rigidity properties are explored for CR maps into hyperquadrics in [BH], [BEH1, BEH2], [EHZ1, EHZ2], [ES], [Ng3], etc (See also [D1], [DL] for irrigidity phenomena in this setting.) The crucial idea to establish Theorem 1.1 is to recognize  $D_m^{IV}$  as an isometric submanifold of the generalized complex unit ball so that we can apply techniques from CR geometry. As an important step to approach Theorem 1.1, we establish a rigidity result first for holomorphic proper maps from the unit ball to the generalized balls. This part is motivated by the framework of Baouendi-Ebenfelt-Huang ([BEH2]) and is proved by a very similar argument.

In another direction, since the work of Bochner [B] and Calabi [C], lots of efforts have been made to understand the local holomorphic isometry  $F : U \rightarrow \Omega_2$  with respect to the (normalized) Bergman metrics of  $\Omega_1$  and  $\Omega_2$  respectively, i.e.  $F^*\omega_{\Omega_2} = \lambda\omega_{\Omega_1}$  on  $U$ , where  $U \subset \Omega_1$  is a connected open set. This problem is largely motivated by the algebraic dynamics problem considered by Clozel-Ullmo [CU] (cf. [M4] [MN] [Yu] [FHX] et al for further developments). Mok ([M4]) proved that  $F$  extends to an algebraic proper holomorphic isometry from  $\Omega_1$  to  $\Omega_2$ . Assume  $\dim_{\mathbb{C}}(\Omega_1) \geq 2$  and  $\Omega_1$  is irreducible. Mok proved that  $F$  is totally geodesic if  $\text{rank}(\Omega_1) \geq 2$  [M4]. When  $\text{rank}(\Omega_1) = 1$ , and  $\Omega_2$  is the product of complex unit balls,  $F$  is also totally geodesic by works of Mok [M2], Ng [Ng2], the second author and Zhang [YZ]. However, when  $\Omega_1 = \mathbb{B}^n$  and  $\Omega_2$  is a bounded symmetric domain other than the product of unit balls, the total geodesy fails dramatically [M5]. In fact, assuming that  $\Omega_2$  is irreducible and  $\text{rank}(\Omega_2) \geq 2$ , Mok constructed a non-totally geodesic holomorphic isometry from  $\mathbb{B}^n$  into  $\Omega$  by using the theory of variety of minimal rational tangents [M5]. In the next theorem, we classify the local holomorphic isometries from  $\mathbb{B}^n$  into  $D_{n+1}^{IV}$ . We refer to [XY] for the study in the general case and note that this problem is studied independently by Chan-Mok in [CM].

**Theorem 1.4.** *Let  $n \geq 2$  and  $F$  be any holomorphic isometry from  $U \subset \mathbb{B}^n$  into  $D_{n+1}^{IV}$  satisfying*

$$F^*\omega_{D_{n+1}^{IV}} = \lambda\omega_{\mathbb{B}^n} \text{ on } U$$

for some  $\lambda > 0$ . Then  $F$  is equivalent to either the map in (1) or the map in (2).

The novelty in Theorem 1.1 is that, as long as the codimension is small, the isometry is implied by properness, which is the converse statement of Mok's theorem ([M4]). This can be applied to obtain the following corollary.

**Corollary 1.5.** *Assume  $n \geq 4, K \geq 1, N_1, \dots, N_K \leq 2n - 3$ . Let  $F = (F_1, \dots, F_K)$  be a holomorphic map from a connected open set  $U \subset \mathbb{B}^n$  into the product of Type IV domains  $\Omega = D_{N_1}^{IV} \times \dots \times D_{N_K}^{IV}$  satisfying the following isometric equation:*

$$\lambda \omega_{\mathbb{B}^n} = \sum_{l=1}^K F_l^* \omega_{D_{N_l}^{IV}} = F^* \omega_{\Omega} \text{ on } U \quad (3)$$

for some positive constant  $\lambda$ . Then each  $F_l, 1 \leq l \leq K$ , is either a constant map or extends to a holomorphic isometric embedding from  $\mathbb{B}^n$  to  $D_{N_l}^{IV}$  with  $F^*(\omega_{D_{N_l}^{IV}}) = \frac{N_l}{n+1} \omega_{\mathbb{B}^n}$ .

We would like to point out that the same conclusion can be made with the slightly more general assumption

$$\omega_{\mathbb{B}^n} = \sum_{l=1}^m \lambda_l F_l^* \omega_{D_{N_l}^{IV}} \text{ on } U$$

instead of (3) for positive constants  $\lambda_l$ . The difference in the proof is that the algebraicity in this case follows from the argument in [HY1, HY2].

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## 2 Preliminaries and Some Examples

Hermitian symmetric manifolds of non-compact type are realized as the four types of Cartan's classical domains and two exceptional cases in complex Euclidean spaces(cf. [H2] [M1]). In particular, the complex unit ball  $\mathbb{B}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 < 1\}$  in  $\mathbb{C}^n$  is a special case of the type I classical domain. The Bergman kernel is given by

$$K_{\mathbb{B}^n}(z, \bar{z}) = c_I (1 - |z|^2)^{-(n+1)}.$$

The type IV classical domain is defined as

$$D_m^{IV} = \{Z = (z_1, \dots, z_m) \in \mathbb{C}^m \mid |Z\bar{Z}^t| < 2 \text{ and } 1 - Z\bar{Z}^t + \frac{1}{4}|ZZ^t|^2 > 0\},$$

where  $Z^t$  is the transpose of  $Z$  and the Bergman kernel function  $K_{D_m^{IV}}(Z, \bar{Z})$  is explicitly given by

$$K_{D_m^{IV}}(Z, \bar{Z}) = c_{IV} \left( 1 - Z\bar{Z}^t + \frac{1}{4}|ZZ^t|^2 \right)^{-m}. \quad (4)$$

Here  $c_I, c_{IV}$  are positive constants depending on the dimensions. (cf. [H2] [M1]). The Bergman metric

$$\omega_\Omega(Z) := \sqrt{-1}\partial\bar{\partial}\log K_\Omega(Z, \bar{Z})$$

of a bounded symmetric domain  $\Omega$  is Kähler-Einstein as the Bergman kernel function is invariant under the holomorphic automorphisms. A straightforward computation shows that  $\partial D_m^{IV} = \{Z \in \mathbb{C}^m : 1 - Z\bar{Z}^t + \frac{1}{4}|ZZ^t|^2 = 0, Z\bar{Z}^t \leq 2\}$ . Moreover, if  $m \geq 2$ , the singular set  $P$  of  $\partial D_m^{IV}$  is given by

$$P = \{Z \in \mathbb{C}^m | 1 - Z\bar{Z}^t + \frac{1}{4}|ZZ^t|^2 = 0, Z\bar{Z}^t = 2\},$$

which is a real  $m$ -dimensional variety.

In [M5], Mok constructed non-totally geodesics holomorphic isometries from the complex unit ball into the irreducible bounded symmetric domain  $\Omega$  when  $\text{rank}(\Omega) \geq 2$  by using the theory of varieties of minimal rational tangents. For the purpose of the current paper, we formulate Mok's theorem merely in the case of Type IV domains.

**Theorem 2.1** (Mok). *Assume  $m \geq 2$ .*

(i) *If  $F : \mathbb{B}^n \rightarrow D_m^{IV}$  is a holomorphic isometry, then  $n \leq m - 1$ .*

(ii) *There exists a non-totally geodesic holomorphic isometric embedding  $G : \mathbb{B}^{m-1} \rightarrow D_m^{IV}$  with  $G^*\omega_{D_m^{IV}} = \lambda\omega_{\mathbb{B}^{m-1}}$  for some  $\lambda > 0$ .*

We now make some remarks on our main theorems. The following simple fact explains why we only consider  $m \geq n + 1$  in our main theorems.

**Lemma 2.2.** *Let  $n \geq 2$  and  $F : \mathbb{B}^n \rightarrow D_m^{IV}$  be a proper holomorphic map. Then  $m \geq n + 1$ .*

*Proof.* It suffices to show that there is no proper holomorphic map for  $m = n \geq 2$ . Suppose that  $F : \mathbb{B}^n \rightarrow D_n^{IV}$  is a proper holomorphic map. Note that  $\mathbb{B}^n$  and  $D_n^{IV}$  are bounded complete circular domains. Then  $F$  is algebraic by Bell's algebraicity result [B]. It then extends holomorphically to a neighborhood  $U$  of an open piece of the boundary  $\partial\mathbb{B}^n$ . Note that the image of  $\partial\mathbb{B}^n \cap U$  cannot be contained in the singular set of  $\partial D_n^{IV}$ . One then easily achieves a contradiction since  $\partial D_n^{IV}$  is Levi-degenerate at any smooth point.  $\square$

**Remark 2.3.** *The statement of Lemma 2.2 fails if  $n = 1$ .  $F(z) = \sqrt{2}z$  is indeed a holomorphic isometry from the unit disc  $\Delta$  to  $D_1^{IV}$ .*

As mentioned in Section 1, the assumption that  $F$  is algebraic in Theorem 1.1 is a necessary condition by Mok's result. We now give an example of non-algebraic proper map from  $\mathbb{B}^n$  to  $D_{n+2}^{IV}$  that is not an isometry.

**Example 2.4.** Assume  $n \geq 2$  and let  $H = (h_1, \dots, h_{n+1}) : \mathbb{B}^n \rightarrow \mathbb{B}^{n+1}$  be a proper holomorphic map with  $H(0) = 0$  such that  $H$  is continuous up to the boundary  $\partial\mathbb{B}^n$  but is not twice continuously differentiable up to any open piece of boundary  $\partial\mathbb{B}^n$  (See [Do]). In particular,  $H$  is not algebraic. Define

$$g = 1 - \sqrt{1 - \sum_{j=1}^{n+1} h_j^2}.$$

Then  $g$  is holomorphic in  $\mathbb{B}^n$  and satisfies

$$g = \frac{1}{2} \left( \sum_{j=1}^{n+1} h_j^2 + g^2 \right).$$

It follows that

$$\sum_{j=1}^{n+1} |h_j|^2 + |g|^2 - \frac{1}{4} \left| \sum_{j=1}^{n+1} h_j^2 + g^2 \right|^2 = \sum_{j=1}^{n+1} |h_j|^2 = 1 \text{ on } \partial\mathbb{B}^n,$$

and thus  $G = (H, g) : \mathbb{B}^n \rightarrow D_{n+2}^{IV}$  is a proper holomorphic map and is not an isometry.

We now recall the definition of CR transversality. Let  $M \subset \mathbb{C}^n, M' \subset \mathbb{C}^N$  be two CR submanifolds. A CR map  $H : M \rightarrow M'$  is called CR transversal at  $p \in M$  if  $dH(\mathcal{C}T_p M)$  is not contained in  $\mathcal{V}'_{F(p)} \oplus \overline{\mathcal{V}'_{F(p)}}$ , where  $\mathcal{V}'$  is the CR bundle of  $M'$ . Example 2.5 shows that the statement of Theorem 1.2 fails if the transversality assumption is dropped, and the statement of Theorem 1.1 fails if the assumption  $F(U) \not\subset \partial D_m^{IV}$  is dropped.

**Example 2.5.** Let  $n \geq 1, m \geq 3$ . Let  $G$  be a holomorphic map from a small neighborhood  $U$  of  $p \in \partial\mathbb{B}^n$  defined by

$$G = \left( \frac{1 + \psi}{\sqrt{2}}, \frac{1 - \psi}{\sqrt{-2}}, 0, \dots, 0 \right),$$

where there are  $m - 2$  zero components and  $\psi$  is any holomorphic function in  $U$ . It is easy to see that  $G$  maps  $U$  to  $\partial D_m^{IV}$  and does not extend to an isometry.

Example 2.6 show that the statement of Theorem 1.1 fails when  $m \geq 2n$ . Example 2.7 shows that the statement of Theorem 1.2 fails if  $n = 1$  and the statement of Theorem 1.1 fails for  $n = 1, m \geq 2$ .

**Example 2.6.** Assume  $n \geq 2$  and let  $H = (h_1, \dots, h_{2n-1}) = (z_1, \dots, z_{n-1}, z_1 z_n, \dots, z_n^2) : \mathbb{B}^n \rightarrow \mathbb{B}^{2n-1}$  be the well-known Whitney map. Define the holomorphic function

$$g = 1 - \sqrt{1 - \sum_{j=1}^{2n-1} h_j^2}$$

on  $\mathbb{B}^n$ . Note that

$$\sum_{i=1}^{2n-1} |h_i|^2 + |g|^2 - \frac{1}{4} \left| \sum_{i=1}^{2n-1} h_i^2 + g^2 \right|^2 = \sum_{i=1}^{n-1} |z_i|^2 + |z_n|^2 \sum_{i=1}^n |z_i|^2.$$

It is easy to verify that  $G = (H, g)$  is a holomorphic polynomial proper map from  $\mathbb{B}^n$  to  $D_{2n}^{IV}$ , while it is not an isometry.

**Example 2.7.** Assume  $k \geq 1$ . Let  $G_k$  be the holomorphic map from the unit disc  $\Delta \subset \mathbb{C}$  to  $D_2^{IV}$  defined by

$$G_k = (z^k, 1 - \sqrt{1 - z^{2k}}). \quad (5)$$

It is easy to verify that  $G_k$  is a proper holomorphic map. Moreover,  $G_k$  is an isometry from  $\Delta$  to  $D_2^{IV}$  if and only if  $k = 1$  (See Proposition 2.11).

Note that all the proper map examples above are constructed from proper maps between balls. It would be reasonable to ask whether all holomorphic proper maps are obtained in this manner. More precisely, let  $H = (h_1, \dots, h_m) : \mathbb{B}^n \rightarrow D_m^{IV}$  be a proper holomorphic map. Does there always exist a holomorphic proper map  $F = (f_1, \dots, f_{m'}) : \mathbb{B}^n \rightarrow \mathbb{B}^{m'}$  with  $m' \leq m$ , such that the following relation holds:

$$\sum_{i=1}^m |h_i|^2 - \frac{1}{4} \left| \sum_{i=1}^m h_i^2 \right|^2 = \sum_{j=1}^{m'} |f_j|^2 \quad ? \quad (6)$$

The answer is negative in general (See Example 2.8), while we suspect that it is true when  $m$  is small compared to  $n$ .

**Example 2.8.** Assume  $n \geq 2$ . Let  $H = (h_1, \dots, h_{4n-1})$  be the proper holomorphic map from  $\mathbb{B}^n$  to  $D_{4n-1}^{IV}$  defined by

$$\left( z_n, \frac{z_1}{\sqrt{2}}, \frac{z_1}{\sqrt{-2}}, \dots, \frac{z_{n-1}}{\sqrt{2}}, \frac{z_{n-1}}{\sqrt{-2}}, \frac{z_1 z_n^2}{2\sqrt{2}}, \frac{z_1 z_n^2}{2\sqrt{-2}}, \dots, \frac{z_{n-1} z_n^2}{2\sqrt{2}}, \frac{z_{n-1} z_n^2}{2\sqrt{-2}}, \frac{z_n^3}{2\sqrt{2}}, \frac{z_n^3}{2\sqrt{-2}} \right).$$

By computation, we have

$$\sum_{i=1}^m |h_i|^2 - \frac{1}{4} \left| \sum_{i=1}^m h_i^2 \right|^2 = \sum_{i=1}^n |z_i|^2 - \frac{1}{4} \left( |z_n|^4 (1 - \sum_{i=1}^n |z_i|^2) \right). \quad (7)$$

Note that the right hand side of (7) cannot be written as sum of norm squares of holomorphic functions (cf. the proof of Proposition 2.11). This implies there is no proper holomorphic map  $F : \mathbb{B}^n \rightarrow \mathbb{B}^{m'}$  such that (6) holds.

On the other hand, it would still be interesting to study how the initial boundary regularity of the map  $F$  can be relaxed in the hypotheses of Theorem 1.2. We make the following conjecture along this line:

**Conjecture 2.9.** *Let  $n \geq 2$ . Any proper holomorphic map from  $\mathbb{B}^n$  to  $D_{n+1}^{IV}$  is a holomorphic isometry (and thus it is equivalent to one of the maps in Theorem 1.4).*

**Remark 2.10.** *Example 2.7 also shows that the statement of Conjecture 2.9 fails when  $n = 1$ .*

We end this section with the following observation on isometric constants. Note that this result is obtained independently by Chan-Mok [CM].

**Proposition 2.11.** *Let  $F : \mathbb{B}^n \rightarrow D_m^{IV}$  be a local holomorphic isometric embedding with respect to the Bergman metrics of isometric constant  $\lambda > 0$ , i.e.*

$$F^* \omega_{D_m^{IV}} = \lambda \omega_{\mathbb{B}^n}. \quad (8)$$

Then

- If  $n \geq 2$ , then  $\lambda = m/(n + 1)$ .
- If  $n = 1$ , then  $\lambda = m/2$  or  $m$ .

*Proof.* Without loss of generality, we may assume  $F(0) = 0$  by composing the automorphisms of  $\mathbb{B}^n$  and  $D_m^{IV}$ . By the standard reduction, (8) is equivalent to

$$(1 - |z|^2)^{\lambda(n+1)/m} = 1 - F(z)\overline{F(z)}^t + \frac{1}{4}|F(z)F(z)^t|^2 \quad (9)$$

that holds in a neighborhood  $U$  of 0. Note that the signature of the left hand side of (9) is either  $(1, s)$  or  $(2, s)$  for some integer  $s \geq 0$ , meaning that it can be written as linear combination of 1 or 2 sum of squares minus  $s$  sum of square of linearly independent holomorphic functions over positive real numbers. In fact, it is of signature  $(1, s)$  if and only if  $F(z)F(z)^t \equiv 0$ . Obviously, the left hand side is of finite rank if and only if  $\lambda(n+1)/m \in \mathbb{N}$  (cf. [Um]). Write  $p = \lambda(n+1)/m$ . We first consider the case  $n = 2$ . Assume  $p \geq 2$ . Applying binomial formula,

$$\left(1 - \sum_{j=1}^n |z_j|^2\right)^p = \sum_{k=0}^p (-1)^k \binom{p}{k} \left(\sum_{j=1}^n |z_j|^2\right)^{p-k}.$$

Note that the monomials on the right hand side are linearly independent and thus the right hand side is of signature  $(r', s')$  with  $r' \geq 3$  if  $n \geq 2$ . Therefore, the left hand side and right hand side of (9) have different signatures and this is a contradiction (cf. [D2] or [Um]). This implies the first part of the Lemma for  $n \geq 2$ . In the case  $n = 1$ , one can similarly get a contradiction if  $p \geq 3$ . The Lemma is then established.  $\square$

**Remark 2.12.** *Both  $\lambda = m/2$  and  $m$  can be obtained when  $n = 1$  in Proposition 2.11. Indeed, the map  $G_1$  in (5) is a holomorphic isometry with isometric constant  $\lambda = 1$  while the isometric map  $F(z) = (\sqrt{2}z, 0) : \Delta \rightarrow D_2^{IV}$  has the isometric constant  $\lambda = 2$ .*

**Remark 2.13.** *The same argument yields that if  $F : \mathbb{B}^n \rightarrow \Omega$  is a holomorphic isometric embedding from a unit ball into a classical symmetric domain with respect to Bergman metric of isometric constant  $\lambda$ , then  $\lambda(n + 1)/(p + q)$ ,  $\lambda(n + 1)/(m - 1)$  or  $\lambda(n + 1)/(m + 1)$  is a positive integer when  $\Omega = D_{p,q}^I$ ,  $D_m^{II}$  or  $D_m^{III}$  respectively.*



### 3 On CR maps from $\partial\mathbb{B}^n$ to $\partial D_m^{IV}$

We will prove Theorem 1.1, Theorem 1.2 and Corollary 1.5 in this section. To this end we first introduce the generalized complex balls and their indefinite metrics.

**Definition 3.1.** *Let  $0 \leq l \leq n, n \geq 2$ .*

$$\mathbb{B}_l^{n+l} = \{(w_1, \dots, w_l, z_1, \dots, z_n) \in \mathbb{C}^{n+l} \mid \sum_{j=1}^n |z_j|^2 - \sum_{j=1}^l |w_j|^2 < 1\}$$

*is called the generalized complex ball with signature  $l$  in  $\mathbb{C}^{n+l}$  and*

$$\omega_{\mathbb{B}_l^{n+l}} = -\sqrt{-1} \partial \bar{\partial} \log \left( 1 + \sum_{j=1}^l |w_j|^2 - \sum_{j=1}^n |z_j|^2 \right)$$

*is the indefinite Kähler metric on  $\mathbb{B}_l^{n+l}$ .*

Note that  $\mathbb{B}_l^{n+l}$  is the indefinite complex hyperbolic space form and  $\mathbb{B}_0^n$  is the complex unit ball  $\mathbb{B}^n$  and  $\omega_{\mathbb{B}_0^n}$  is the Bergman metric on  $\mathbb{B}^n$  up to a positive constant  $n+1$ . The holomorphic automorphism group of  $\mathbb{B}_l^{n+l}$  can be described as follows. We first embed  $\mathbb{B}_l^{n+l}$  into  $\mathbb{P}^{n+l}$  as an open set by standard coordinate embedding:

$$(w_1, \dots, w_l, z_1, \dots, z_n) \rightarrow [1, w_1, \dots, w_l, z_1, \dots, z_n].$$

Therefore  $\mathbb{B}_l^{n+l} = \{[s, w_1, \dots, w_l, z_1, \dots, z_n] \in \mathbb{P}^{n+l} \mid s \neq 0, |s|^2 + \sum_{j=1}^l |w_j|^2 - \sum_{j=1}^n |z_j|^2 > 0\}$  in homogenous coordinates. The holomorphic automorphism group of  $\mathbb{B}_l^{n+l}$  is a subgroup of the holomorphic automorphism group of  $\mathbb{P}^{n+l}$  given by

$$U(n+l+1, l+1) = \{A \in GL(n+l+1, \mathbb{C}) \mid AE(l+1, n+l+1)\bar{A}^t = E(l+1, n+l+1)\},$$

where  $E(l+1, n+l+1)$  is a diagonal matrix with  $-1$  on the first  $l+1$  diagonal elements and then  $1$  on the next  $n$  diagonal elements. It is easy to check that  $\omega_{\mathbb{B}_l^{n+l}}$  is invariant under  $U(n+l+1, l+1)$ .

The connection between classical domain of Type IV and the generalized complex balls is as follows. Define  $L : D_m^{IV} \rightarrow \mathbb{B}_1^{m+1}$  by  $L(z_1, \dots, z_m) = [1, \frac{1}{2} \sum_{j=1}^m z_j^2, z_1, \dots, z_m]$ . One can easily check that  $L$  is a proper holomorphic map and furthermore is a holomorphic isometric embedding up to a positive constant  $1/m$ , i.e.  $L^* \omega_{\mathbb{B}_1^{m+1}} = \frac{1}{m} \omega_{D_m^{IV}}$ . The following is a key result in proving Theorem 1.1.

**Theorem 3.2.** *Assume  $n \geq 4, n+1 \leq N \leq 2n-2$  and assume  $U$  is a connected open set in  $\mathbb{C}^n$  containing  $p \in \partial\mathbb{B}^n$ ,  $V$  is an open set in  $\mathbb{C}^N$  containing  $q \in \partial\mathbb{B}_1^N$ . Let  $F$  be a holomorphic map on  $U$  such that*

- $F(U) \not\subset \partial \mathbb{B}_1^N$ ,
- $F(U \cap \partial \mathbb{B}_1^n) \subset \partial \mathbb{B}_1^N$ .

Then the following statement holds:

- If  $N = n + 1$ , then  $F$  is equivalent to

$$(0, z_1, \dots, z_n) \tag{10}$$

- If  $N > n + 1$ , then  $F$  is equivalent to

$$(\psi, \psi, z_1, \dots, z_n, \mathbf{0}), \tag{11}$$

where  $\mathbf{0}$  denotes the  $(N - n - 2)$ -dimensional zero row vector and  $\psi$  is some holomorphic function on  $U$ .

The proof follows from a very similar argument as in [BEH2]. The crucial techniques in the proof include the normal form type argument, Huang's Lemma [Hu1] and its generalizations [BEH2], moving point trick [Hu1], and a transversality result in [BER2]. To make it easier for the readers, we will sketch a proof in Appendix. The following corollary follows from Theorem 3.2.

**Corollary 3.3.** *Assume the same assumptions as in Theorem 3.2. Then  $F$  is a local holomorphic isometric embeddings from  $\mathbb{B}^n$  to  $\mathbb{B}_1^N$ , i.e.  $F^* \omega_{\mathbb{B}_1^N} = \omega_{\mathbb{B}^n} / (n + 1)$  on  $U \cap \mathbb{B}^n$ .*

*Proof.* It is straightforward to check that both maps in (10) and (11) are isometries from  $\mathbb{B}^n$  to  $\mathbb{B}_1^N$ , i.e.  $F^* \omega_{\mathbb{B}_1^N} = \omega_{\mathbb{B}^n} / (n + 1)$ . Then Theorem 3.3 follows directly from Theorem 3.2 as  $\omega_{\mathbb{B}^n}$  and  $\omega_{\mathbb{B}_1^N}$  are invariant under automorphisms.  $\square$

We are now at the position to prove Theorem 1.1, 1.2 and Corollary 1.5. Theorem 1.4 will be proved in the next section.

**Proof of Theorem 1.1.** The composition  $L \circ F$  is a holomorphic map satisfying the assumption in Corollary 3.3. Thus  $L \circ F$  is a local holomorphic isometry from  $\mathbb{B}^n$  to  $\mathbb{B}_1^{m+1}$  by Corollary 3.3, i.e.  $(L \circ F)^* \omega_{\mathbb{B}_1^{m+1}} = \omega_{\mathbb{B}^n}$  on  $U \cap \mathbb{B}^n$ . Note that

$$\omega_{\mathbb{B}^n} / (n + 1) = (L \circ F)^* \omega_{\mathbb{B}_1^{m+1}} = F^* \left( L^* \omega_{\mathbb{B}_1^{m+1}} \right) = \frac{1}{m} F^* \omega_{D_m^{IV}} \text{ on } U \cap \mathbb{B}^n.$$

Therefore  $F$  extends to a global holomorphic isometry from  $\mathbb{B}^n$  to  $D_m^{IV}$  by Mok's theorem [M4]. We conclude  $F^* \omega_{D_m^{IV}} = \frac{m}{n+1} \omega_{\mathbb{B}^n}$  by Proposition 2.11.  $\square$

**Proof of Theorem 1.2 assuming Theorem 1.4.** An easy computation verifies that  $\partial D_{n+1}^{IV}$  is holomorphically nondegenerate (See [BER1]) at any smooth point. Indeed,  $\partial D_{n+1}^{IV}$  is 2-nondegenerate in the sense of Baouendi-Huang-Rothschild [BHR] at any smooth point. It

follows from a regularity result in [KLX] that  $F$  is real analytic in some open piece  $V$  of  $\partial\mathbb{B}^n$ .  $F$  thus extends holomorphically to a neighborhood of  $V$ . We conclude that  $F$  is an isometry by Theorem 1.1. Then Theorem 1.2 follows from Theorem 1.4.

**Proof of Corollary 1.5.** First we note that by the algebraic extension theorem of Mok [M4] (following from the argument in [HY1, HY2] in this case as well),  $F$  is an algebraic map. By holomorphic continuation,  $F$  can be extended along some path to an open set  $V$  in  $\mathbb{C}^n$  containing a generic boundary point of  $\mathbb{B}^n$  and the holomorphic isometry equation (3) is preserved along the path. Therefore, there must exist a nonconstant holomorphic map  $F_l$  such that  $F(V \cap \partial\mathbb{B}^n) \subset \partial D_{N_l}^{IV}$  and  $F(V) \not\subset \partial D_{N_l}^{IV}$ . It follows from Theorem 1.1 that  $F_l$  is a holomorphic isometric embedding. The corollary then follows from an induction argument.

## 4 On holomorphic isometries from $\mathbb{B}^n$ into $D_{n+1}^{IV}$

### 4.1 Examples of holomorphic isometries

Write  $z = (z_1, \dots, z_n)$  to be the coordinates in  $\mathbb{C}^n$  for  $n \geq 2$ . Let  $R_n^{IV} : \mathbb{B}^n \rightarrow D_{n+1}^{IV}$  be defined as

$$R_n^{IV} = (f_1, \dots, f_{n-1}, f_n, f_{n+1}), \quad (12)$$

where  $f_i = z_i, 1 \leq i \leq n-1, f_n = \frac{P_n}{Q}, f_{n+1} = \frac{P_{n+1}}{Q}$ ,

$$P_n = \frac{1}{2} \sum_{i=1}^{n-1} z_i^2 - z_n^2 + z_n, P_{n+1} = -\sqrt{-1} \left( \frac{1}{2} \sum_{i=1}^{n-1} z_i^2 + z_n^2 - z_n \right), Q = \sqrt{2}(1 - z_n).$$

It is easy to verify that

$$\sum_{i=1}^n f_i^2 = \frac{\sum_{i=1}^{n-1} z_i^2}{1 - z_n}, \quad |f_n|^2 + |f_{n+1}|^2 = \frac{1}{4} \frac{|\sum_{i=1}^{n-1} z_i^2|^2}{|1 - z_n|^2} + |z_n|^2.$$

It then follows that  $R_n^{IV}$  is a holomorphic isometry from  $\mathbb{B}^n$  to  $D_{n+1}^{IV}$ . In fact, we will show that  $R_n^{IV}$  is the unique rational holomorphic isometry from  $\mathbb{B}^n$  to  $D_{n+1}^{IV}$  up to holomorphic automorphisms.

For any  $\theta \in [0, \pi/4)$ , define  $h_\theta(z) = 1 + 2\sqrt{-1} \sin(2\theta)z_n - z_n^2 - \cos(2\theta) \left( \sum_{j=1}^{n-1} z_j^2 \right)$ . Let  $I_{n,\theta} = (f_1, \dots, f_{n+1})$  be

$$\begin{aligned} f_1(z) &= z_1, \dots, f_{n-1}(z) = z_{n-1}; \\ f_n(z) &= \frac{(\cos \theta + \sqrt{-1} \sin \theta z_n) - \cos \theta \sqrt{h_\theta(z)}}{\cos(2\theta)}; \\ f_{n+1}(z) &= \frac{(-\sqrt{-1} \sin \theta + \cos \theta z_n) + \sqrt{-1} \sin \theta \sqrt{h_\theta(z)}}{\cos(2\theta)}. \end{aligned} \quad (13)$$

Then  $I_{n,\theta}$  is a holomorphic isometry from  $\mathbb{B}^n$  to  $D_{n+1}^{IV}$ . In particular,

$$I_{n,0} = \left( z_1, \dots, z_{n-1}, 1 - \sqrt{1 - \sum_{j=1}^n z_j^2}, z_n \right). \quad (14)$$

We will also show that  $I_{n,0}$  is the unique irrational holomorphic isometry from  $\mathbb{B}^n$  to  $D_{n+1}^{IV}$  up to holomorphic automorphisms.

## 4.2 Singularities of holomorphic isometries

The rational holomorphic isometry  $R_n^{IV}$  given in previous sections is not a totally geodesic and only produces singularities at one single point on the boundary  $\partial\mathbb{B}^n$ . When  $n \geq 2$ , one can easily avoid passing through this point by slicing  $\mathbb{B}^n$  with a complex linear hyperplane. Therefore, one obtains a holomorphic polynomial isometry from  $\mathbb{B}^{n-1}$  into  $D_{n+1}^{IV}$ . In particular, this answers the question raised by Mok in [M3] (Question 5.2.2) in the negative for Type IV domains and counter-examples for other types of classical domains are given in [XY]. Note that this type of examples are discovered independently by Chan-Mok [CM].

**Theorem 4.1.** *Assume  $m \geq n + 2$ . There exist non-totally geodesic holomorphic isometries from the unit ball  $\mathbb{B}^n$  into  $D_m^{IV}$  that can be extended holomorphically to  $\mathbb{C}^n$ .*

*Proof.* The map can be indeed chosen to be polynomial. The polynomial holomorphic isometries is given by

$$\left( z_1, \dots, z_n, \frac{\sqrt{2}}{4} \sum_{i=1}^n z_i^2, \frac{\sqrt{-2}}{4} \sum_{i=1}^n z_i^2, 0, \dots, 0 \right) : \mathbb{B}^n \rightarrow D_m^{IV} \text{ for } m \geq n + 2, n \geq 1,$$

where there are  $(m - n - 2)$  zero components in the map. □

## 4.3 Proof of Theorem 1.4

This subsection is devoted to establish Theorem 1.4. For that we will first describe the holomorphic automorphism group action on  $D_m^{IV}$  in terms of the Borel embedding (cf. [H1] [M1]). The hyperquadric  $\mathbb{Q}^m$ , the compact dual of  $D_n^{IV}$  is defined by  $\mathbb{Q}^m := \{[z_1, \dots, z_{m+2}] \in \mathbb{P}^{m+1} | z_1^2 + \dots + z_m^2 = z_{m+1}^2 + z_{m+2}^2\}$ . The Borel embedding  $D_m^{IV} \subset \mathbb{Q}^m \subset \mathbb{P}^{m+1}$  is given by

$$Z = (z_1, \dots, z_m) \rightarrow \left[ z_1, \dots, z_m, \frac{1 + \frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}ZZ^t}{\sqrt{-2}} \right].$$

The holomorphic automorphism group of  $D_m^{IV}$  is given by

$$\text{Aut}(D_m^{IV}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in O(m, 2, \mathbb{R}) | \det(D) > 0, \right\}$$

where  $A \in M(m, m, \mathbb{R})$ ,  $B \in M(m, 2, \mathbb{R})$ ,  $C \in M(2, m, \mathbb{R})$ ,  $D \in M(2, 2, \mathbb{R})$ . The automorphism group action is given in the following explicit way. Let  $Z = (z_1, \dots, z_m) \in D_m^{IV}$  and  $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Aut}(D_n^{IV})$ . Write  $Z' = \left( \frac{1+\frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1-\frac{1}{2}ZZ^t}{\sqrt{-2}} \right)$ . Then the action of  $T$  on  $D_m^{IV}$  is given by

$$T(Z) = \frac{ZA + Z'C}{(ZB + Z'D) \left( 1/\sqrt{2}, \sqrt{-1/2} \right)^t}.$$

Rephrasing in homogenous coordinates, if the holomorphic automorphism maps  $Z = (z_1, \dots, z_m) \in D_m^{IV}$  to  $W = (w_1, \dots, w_m) \in D_m^{IV}$ , then there exists  $T \in \text{Aut}(D_m^{IV})$  such that

$$\left[ w_1, \dots, w_m, \frac{1 + \frac{1}{2}WW^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}WW^t}{\sqrt{-2}} \right] = \left[ z_1, \dots, z_m, \frac{1 + \frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}ZZ^t}{\sqrt{-2}} \right] \cdot T.$$

In other words, there exists nonzero  $\lambda \in \mathbb{C}$ , such that

$$\left( w_1, \dots, w_m, \frac{1 + \frac{1}{2}WW^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}WW^t}{\sqrt{-2}} \right) = \lambda \left( z_1, \dots, z_m, \frac{1 + \frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}ZZ^t}{\sqrt{-2}} \right) \cdot T.$$

Note that the isotropy group  $K_0$  at the origin is  $K_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in O(m, 2, \mathbb{R}) \mid \det(D) = 1 \right\} \cong O(m, \mathbb{R}) \times SO(2, \mathbb{R})$ .

We are now at the position to prove Theorem 1.4. We first establish a result on isotropy equivalence.

**Theorem 4.2.** *Let  $F : \mathbb{B}^n \rightarrow D_{n+1}^{IV}$  be a holomorphic isometric embedding satisfying  $F(0) = 0$  and*

$$F^* \omega_{D_{n+1}^{IV}} = \omega_{\mathbb{B}^n}. \quad (15)$$

*Then  $F$  is isotropically equivalent to either the map  $R_n^{IV}$  in (12) or the map  $I_{n,\theta}$  in (13) for some  $\theta \in [0, \pi/4)$ .*

*Proof. First normalization:* Write  $F = (f_1, \dots, f_{n+1})$ . By the isometry assumption, a standard reduction yields that

$$\sum_{i=1}^{n+1} |f_i|^2 - \frac{1}{4} \left| \sum_{i=1}^{n+1} f_i^2 \right|^2 = \sum_{i=1}^n |z_i|^2. \quad (16)$$

We then have

$$\left( z_1, \dots, z_n, \frac{1}{2} \sum_{j=1}^m f_j^2(z) \right) = (f_1(z), \dots, f_{n+1}(z)) \cdot \mathbf{U} \quad (17)$$

for some  $(n+1) \times (n+1)$  unitary matrix  $\mathbf{U}$  by a lemma of D'Angelo ([D2]). Write  $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_{n+1})$ , where each  $\mathbf{u}_i$  for  $1 \leq i \leq n+1$ , is a column vector in  $\mathbb{C}^{n+1}$ . Write the first  $n$ -columns of  $U$  as  $\mathbf{U}_0 = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ . By (17), we have

$$(z_1, \dots, z_n) = (f_1(z), \dots, f_n(z)) \cdot \mathbf{U}_0. \quad (18)$$

By the singular value decomposition of symmetric matrices, there exists an unitary  $n \times n$  matrix  $\mathbf{V}$  such that

$$\mathbf{V}^t \mathbf{U}_0^t \mathbf{U}_0 \mathbf{V} = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

for real numbers  $\lambda_i$  satisfying  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ . Apply the unitary change of coordinate in  $\mathbb{C}^n$  by letting  $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_n) = z \cdot \mathbf{V}$ , where  $z = (z_1, \dots, z_n)$  or equivalently,  $z = \tilde{z} \cdot \mathbf{V}^{-1}$ . By (18), we have

$$(\tilde{z}_1, \dots, \tilde{z}_n) = (f_1(\tilde{z}\mathbf{V}^{-1}), \dots, f_{n+1}(\tilde{z}\mathbf{V}^{-1})) \hat{\mathbf{U}}_0,$$

where  $\hat{\mathbf{U}}_0 = \mathbf{U}_0 \mathbf{V}$  with  $\hat{\mathbf{U}}_0^t \hat{\mathbf{U}}_0 = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ . Moreover, by (17), we have

$$\left( \tilde{z}_1, \dots, \tilde{z}_n, \frac{1}{2} \sum_{i=1}^{n+1} f_i^2(\tilde{z}\mathbf{V}^{-1}) \right) = (f_1(\tilde{z}\mathbf{V}^{-1}), \dots, f_{n+1}(\tilde{z}\mathbf{V}^{-1})) (\hat{\mathbf{U}}_0, \mathbf{u}_{n+1}). \quad (19)$$

Let  $\tilde{F}(\tilde{z}) = F(\tilde{z}\mathbf{V}^{-1})$  and write  $\tilde{F}(\tilde{z}) = (\tilde{f}_1(\tilde{z}), \dots, \tilde{f}_{n+1}(\tilde{z}))$ . Then equation (19) can be rewritten as

$$\left( \tilde{z}_1, \dots, \tilde{z}_n, \frac{1}{2} \sum_{i=1}^{n+1} \tilde{f}_i^2(\tilde{z}) \right) = (\tilde{f}_1(\tilde{z}), \dots, \tilde{f}_{n+1}(\tilde{z})) (\hat{\mathbf{U}}_0, \mathbf{u}_{n+1}). \quad (20)$$

Note that  $(\hat{\mathbf{U}}_0, \mathbf{u}_{n+1})$  is still an unitary  $(n+1) \times (n+1)$  matrix. It follows from (20) that  $\tilde{F}$  is also a holomorphic isometry from  $\mathbb{B}^n$  to  $D_{n+1}^{IV}$  and moreover,  $\tilde{F}$  is equivalent to  $F$ .

**Second normalization:** In the following, we write  $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^k x_i y_i$  for two  $k$ -dimensional vectors  $\mathbf{x} = (x_1, \dots, x_k)$ ,  $\mathbf{y} = (y_1, \dots, y_k)$ . We now consider the new map  $\tilde{F}$  in the new holomorphic coordinate  $\tilde{z}$ . But for the simplicity of notations, we still use  $F, z, \mathbf{U}_0$  to denote  $\tilde{F}, \tilde{z}, \hat{\mathbf{U}}_0$  respectively. Therefore, we have

$$\left( z_1, \dots, z_n, \frac{1}{2} \sum_{j=1}^m f_j^2(z) \right) = (f_1(z), \dots, f_{n+1}(z)) \cdot (\mathbf{U}_0, \mathbf{u}_{n+1})$$

with

$$\mathbf{U}_0^t \mathbf{U}_0 = \text{diag}\{\lambda_1, \dots, \lambda_n\}. \quad (21)$$

Write  $\mathbf{U}_0 = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ . It follows from (21) that

$$\begin{aligned} \mathbf{u}_i \cdot \mathbf{u}_i &= \lambda_i \in \mathbb{R} \text{ for } 1 \leq i \leq n, \\ \mathbf{u}_i \cdot \mathbf{u}_j &= 0 \text{ for } 1 \leq i \neq j \leq n. \end{aligned} \quad (22)$$

Note  $\bar{\mathbf{U}}_0^t \mathbf{U}_0 = \mathbf{I}_n$ . We have

$$\begin{aligned} \mathbf{u}_i \cdot \bar{\mathbf{u}}_i &= 1 \text{ for } 1 \leq i \leq n, \\ \mathbf{u}_i \cdot \bar{\mathbf{u}}_j &= 0 \text{ for } 1 \leq i \neq j \leq n. \end{aligned} \quad (23)$$

Write  $\mathbf{u}_i = \mathbf{a}_i + \sqrt{-1}\mathbf{b}_i$  for  $1 \leq i \leq n$ . It follows from (22)-(23) that

$$\begin{aligned} \mathbf{a}_i \cdot \mathbf{b}_j &= 0 \text{ for } 1 \leq i, j \leq n, \\ \mathbf{a}_i \cdot \mathbf{a}_j &= 0 \text{ for } 1 \leq i \neq j \leq n, \\ \mathbf{b}_i \cdot \mathbf{b}_j &= 0 \text{ for } 1 \leq i \neq j \leq n, \\ \mathbf{a}_i \cdot \mathbf{a}_i + \mathbf{b}_i \cdot \mathbf{b}_i &= 1 \text{ for } 1 \leq i \leq n. \end{aligned} \tag{24}$$

Therefore, these  $2n$  vectors  $\{\mathbf{a}_i, \mathbf{b}_i\}_{1 \leq i \leq n}$  in  $\mathbb{R}^{n+1}$  are mutually orthogonal. This implies that at least  $n-1$  of them are zero vectors. However, by the last equation in (24),  $\mathbf{a}_i$  and  $\mathbf{b}_i$  cannot be both zero for each  $1 \leq i \leq n$ . Hence, by applying again the unitary change of coordinates in  $\mathbb{C}^n$  if necessary, we assume that for each  $1 \leq j \leq n-1$ , either  $\mathbf{a}_j$  or  $\mathbf{b}_j$  is zero. Furthermore, for each fixed  $1 \leq j \leq n-1$ , by applying the unitary change of coordinates

$$(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n) \rightarrow (z_1, \dots, z_{j-1}, \sqrt{-1}z_j, z_{j+1}, \dots, z_n)$$

in  $\mathbb{C}^n$  if necessary, we can always assume that  $\mathbf{b}_j = 0, \mathbf{a}_j \neq 0$  for  $1 \leq j \leq n-1$ . Therefore, we have  $\mathbf{u}_j = \mathbf{a}_j \in \mathbb{R}^{n+1}$  for all  $1 \leq j \leq n-1$ , and moreover,

$$(\mathbf{u}_1, \dots, \mathbf{u}_{n-1})^t (\mathbf{u}_1, \dots, \mathbf{u}_{n-1}) = \mathbf{I}_{n-1}.$$

Extend  $\{\mathbf{u}_1, \dots, \mathbf{u}_{n-1}\}$  to an orthonormal basis  $\{\mathbf{u}_1, \dots, \mathbf{u}_{n-1}, \mathbf{c}_n, \mathbf{c}_{n+1}\}$  of  $\mathbb{R}^{n+1}$  and write the  $(n+1) \times (n+1)$  matrix

$$\mathbf{C} = (\mathbf{u}_1, \dots, \mathbf{u}_{n-1}, \mathbf{c}_n, \mathbf{c}_{n+1}).$$

It follows that  $\mathbf{C}, \mathbf{C}^t$  are orthogonal matrices  $SO(n+1)$  as  $\mathbf{C}\mathbf{C}^t = \mathbf{C}^t\mathbf{C} = \mathbf{I}_{n+1}$ . Define  $\tilde{F} = (\tilde{f}_1, \dots, \tilde{f}_{n+1}) = F \cdot (\mathbf{C}^t)^{-1}$  or equivalently  $F = \tilde{F} \cdot \mathbf{C}^t$ . One can easily check that  $\tilde{F}$  is still a holomorphic isometry from  $\mathbb{B}^n$  into  $D_{n+1}^{IV}$  and  $\tilde{F}$  is equivalent to  $F$ . Furthermore, one has

$$\left( z_1, \dots, z_n, \frac{1}{2} \sum_{j=1}^{n+1} \tilde{f}_j^2 \right) = (\tilde{f}_1, \dots, \tilde{f}_{n+1}) \mathbf{C}^t (\mathbf{U}_0, \mathbf{u}_{n+1}).$$

Since  $\mathbf{C}$  is an orthogonal matrix, then

$$\mathbf{C}^t \mathbf{U}_0 = (\mathbf{X}, \tilde{\mathbf{u}}_n)_{(n+1) \times n},$$

where  $\mathbf{X} = \begin{bmatrix} \mathbf{I}_{n-1} \\ \mathbf{0}_{2 \times (n-1)} \end{bmatrix}$  and  $\mathbf{0}_{2 \times (n-1)}$  is the  $2 \times (n-1)$  zero matrix. Note

$$\overline{(\mathbf{C}^t \mathbf{U}_0)^t} (\mathbf{C}^t \mathbf{U}_0) = \overline{\mathbf{U}}_0^t \mathbf{U}_0 = \mathbf{I}_n,$$

i.e. the columns of  $\mathbf{C}^t \mathbf{U}_0$  are orthonormal basis in  $\mathbb{C}^{n+1}$ . Then we conclude that

$$\tilde{\mathbf{u}}_n = [0, \dots, 0, \xi_1, \xi_2]^t$$

for  $\xi_1, \xi_2 \in \mathbb{C}$  with  $|\xi_1|^2 + |\xi_2|^2 = 1$ .

Note that  $\mathbf{C}^t(\mathbf{U}_0, \mathbf{u}_{n+1})$  is an  $(n+1) \times (n+1)$  unitary matrix. We will again use  $F, \mathbf{U}$  to denote  $\tilde{F}, \mathbf{C}^t(\mathbf{U}_0, \mathbf{u}_{n+1})$  respectively for the simplicity of notations. To summarize the above, we have normalized the original holomorphic isometry to the map  $F$  satisfying

$$\left( z_1, \dots, z_n, \frac{1}{2} \sum_{j=1}^{n+1} f_j^2(z) \right) = (f_1(z), \dots, f_{n+1}(z)) \mathbf{U}, \quad (25)$$

where  $\mathbf{U}$  is  $(n+1) \times (n+1)$  unitary matrix and

$$\mathbf{U} = (\mathbf{X}, \mathbf{u}_n, \mathbf{u}_{n+1})$$

$$\text{for } \mathbf{X} = \begin{bmatrix} \mathbf{I}_{n-1} \\ \mathbf{0}_{2 \times (n-1)} \end{bmatrix}, \mathbf{u}_n = \begin{bmatrix} 0 \\ \dots \\ 0 \\ \xi_1 \\ \xi_2 \end{bmatrix} \text{ and } \mathbf{u}_{n+1} = \begin{bmatrix} 0 \\ \dots \\ 0 \\ \eta_1 \\ \eta_2 \end{bmatrix} \text{ with}$$

$$|\xi_1|^2 + |\xi_2|^2 = 1, |\eta_1|^2 + |\eta_2|^2 = 1 \text{ and } \xi_1 \bar{\eta}_1 + \xi_2 \bar{\eta}_2 = 0.$$

Now Replacing  $F$  by  $\tilde{F} = e^{-\sqrt{-1}\alpha} F$  and writing  $\tilde{F} = (\tilde{f}_1, \dots, \tilde{f}_{n+1})$ , we have:

$$\left( z_1, \dots, z_n, \frac{1}{2} \sum_{j=1}^{n+1} \tilde{f}_j^2(z) \right) = (\tilde{f}_1(z), \dots, \tilde{f}_{n+1}(z)) \cdot \begin{bmatrix} e^{\sqrt{-1}\alpha} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & e^{\sqrt{-1}\alpha} \xi_1 & e^{-\sqrt{-1}\alpha} \eta_1 \\ \mathbf{0}_{(n-1) \times 1}^t & e^{\sqrt{-1}\alpha} \xi_2 & e^{-\sqrt{-1}\alpha} \eta_2 \end{bmatrix}.$$

Choose a suitable  $\alpha$  such that the real and imaginary parts of  $(e^{-\sqrt{-1}\alpha} \eta_1, e^{-\sqrt{-1}\alpha} \eta_2)^t$  are orthogonal. Applying the unitary change of coordinates  $\hat{z} = (\hat{z}_1, \dots, \hat{z}_n) = e^{-\sqrt{-1}\alpha} (z_1, \dots, z_n)$  and defining  $\hat{F}(\hat{z}) = (\hat{f}_1(\hat{z}), \dots, \hat{f}_{n+1}(\hat{z})) = \tilde{F}(e^{\sqrt{-1}\alpha} \hat{z})$ , then one can easily check that  $\hat{F}$  satisfies:

$$\left( \hat{z}_1, \dots, \hat{z}_n, \frac{1}{2} \sum_{j=1}^{n+1} \hat{f}_j^2(\hat{z}) \right) = (\hat{f}_1(\hat{z}), \dots, \hat{f}_{n+1}(\hat{z})) \cdot \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & \xi_1 & e^{-\sqrt{-1}\alpha} \eta_1 \\ \mathbf{0}_{(n-1) \times 1}^t & \xi_2 & e^{-\sqrt{-1}\alpha} \eta_2 \end{bmatrix}. \quad (26)$$

We will still use  $F, z, \eta_i$  to denote  $\hat{F}, \hat{z}, e^{-\sqrt{-1}\alpha} \eta_i$  for  $i = 1, 2$  respectively. Then (26) reads

$$\left( z_1, \dots, z_n, \frac{1}{2} \sum_{j=1}^{n+1} f_j^2(z) \right) = (f_1(z), \dots, f_{n+1}(z)) \cdot \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & \xi_1 & \eta_1 \\ \mathbf{0}_{(n-1) \times 1}^t & \xi_2 & \eta_2 \end{bmatrix}$$



with

$$|\xi_1|^2 + |\xi_2|^2 = 1, |\eta_1|^2 + |\eta_2|^2 = 1 \text{ and } \xi_1\bar{\eta}_1 + \xi_2\bar{\eta}_2 = 0, \operatorname{Re}(\eta_1, \eta_2)^t \perp \operatorname{Im}(\eta_1, \eta_2)^t. \quad (27)$$

By applying an automorphism

$$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 2} \\ \mathbf{0}_{2 \times (n-1)} & \mathbf{V}_{2 \times 2} \end{bmatrix}$$

of  $D_{n+1}^{IV}$  with a suitable  $\mathbf{V} \in O(2)$ , we can further make the real part of vector  $(\eta_1, \eta_2)$  be of form  $(c, 0)$  for some  $c \in \mathbb{R}$ . Consequently, by (27) we conclude that  $\eta_1 \in \mathbb{R}$  and  $\eta_2 = \sqrt{-1}\eta$  for some  $\eta \in \mathbb{R}$ . By further applying an automorphism

$$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{1 \times (n-1)} & \pm 1 & 0 \\ \mathbf{0}_{1 \times (n-1)} & 0 & \pm 1 \end{bmatrix}$$

of  $D_{n+1}^{IV}$  if necessary, we can assume that  $\eta_1 \geq 0$  and  $\eta \geq 0$ . Since  $\eta_1^2 + \eta^2 = |\eta_1|^2 + |\eta_2|^2 = 1$ , write  $\eta_1 = \cos \theta, \eta = \sin \theta$  for  $\theta \in [0, \pi/2]$  and then  $\eta_2 = \sqrt{-1} \sin \theta$ . As  $|\xi_1|^2 + |\xi_2|^2 = 1, \xi_1\bar{\eta}_1 + \xi_2\bar{\eta}_2 = 0$ , write  $\xi_1 = \sqrt{-1} \sin \theta e^{\sqrt{-1}\alpha}, \xi_2 = \cos \theta e^{\sqrt{-1}\alpha}$  for  $\alpha \in [0, 2\pi)$ . By applying an unitary transform  $(\tilde{z}_1, \dots, \tilde{z}_{n-1}, \tilde{z}_n) = (z_1, \dots, z_{n-1}, e^{\sqrt{-1}\alpha} z_n)$  in  $\mathbb{C}^n$  if necessary, we may let  $\xi_1 = \sqrt{-1} \sin \theta, \xi_2 = \cos \theta$ . Therefore, we have normalized the map  $F = (f_1, \dots, f_{n+1})$  to make it satisfy

$$\left( z_1, \dots, z_n, \frac{1}{2} \sum_{j=1}^{n+1} f_j^2(z) \right) = (f_1(z), \dots, f_{n+1}(z)) \cdot \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & \sqrt{-1} \sin \theta & \cos \theta \\ \mathbf{0}_{(n-1) \times 1}^t & \cos \theta & \sqrt{-1} \sin \theta \end{bmatrix}$$

for  $\theta \in [0, \pi/2]$ . Denote the matrix

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & \sqrt{-1} \sin \theta & \cos \theta \\ \mathbf{0}_{(n-1) \times 1}^t & \cos \theta & \sqrt{-1} \sin \theta \end{bmatrix} \quad (28)$$

We the proceed in two different cases.

**Case I:** If  $\theta = \pi/4$ . In this case, we have

$$\left( z_1, \dots, z_n, \frac{1}{2} \sum_{j=1}^{n+1} f_j^2(z) \right) = (f_1(z), \dots, f_{n+1}(z)) \cdot \mathbf{U},$$

where

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & \frac{\sqrt{-2}}{2} & \frac{\sqrt{2}}{2} \\ \mathbf{0}_{(n-1) \times 1}^t & \frac{\sqrt{2}}{2} & \frac{\sqrt{-2}}{2} \end{bmatrix}.$$

By replacing  $F$  by

$$F \cdot \begin{bmatrix} -\sqrt{-1}\mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{1 \times (n-1)} & \sqrt{-1} & 0 \\ \mathbf{0}_{1 \times (n-1)} & 0 & -\sqrt{-1} \end{bmatrix},$$

and then apply the unitary transformation in  $\mathbb{C}^n : (\tilde{z}_1, \dots, \tilde{z}_n) = (-\sqrt{-1}z_1, \dots, -\sqrt{-1}z_{n-1}, \sqrt{-1}z_n)$ , we are able to make

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & -\frac{\sqrt{-2}}{2} & \frac{\sqrt{-2}}{2} \\ \mathbf{0}_{(n-1) \times 1}^t & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

By solving this linear system, one obtains the map  $R_n^{IV}$  in (12). We then replace  $(f_1, \dots, f_{n+1})$  by  $(f_1, \dots, f_n, -f_{n+1})$  to obtain the map in (1).

**Case II:** If  $\theta \in [0, \pi/2]$  with  $\theta \neq \pi/4$ . The conclusion is trivial if  $\theta \in [0, \pi/4)$ . In the sequel, we assume  $\theta \in (\pi/4, \pi/2]$ . Write  $\beta = \pi/2 - \theta \in [0, \pi/4)$ . Then

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & \sqrt{-1} \cos \beta & \sin \beta \\ \mathbf{0}_{(n-1) \times 1}^t & \sin \beta & \sqrt{-1} \cos \beta \end{bmatrix}.$$

By applying the automorphism

$$(\tilde{w}_1, \dots, \tilde{w}_{n+1}) = (w_1, \dots, w_{n+1}) \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & 0 & 1 \\ \mathbf{0}_{(n-1) \times 1}^t & 1 & 0 \end{bmatrix}$$

of  $D_{n+1}^{IV}$ , we may let

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & \sin \beta & \sqrt{-1} \cos \beta \\ \mathbf{0}_{(n-1) \times 1}^t & \sqrt{-1} \cos \beta & \sin \beta \end{bmatrix}.$$

Applying the automorphism  $(\tilde{w}_1, \dots, \tilde{w}_{n+1}) = -\sqrt{-1}(w_1, \dots, w_{n+1})$  of  $D_{n+1}^{IV}$  and then applying the unitary transform  $(\tilde{z}_1, \dots, \tilde{z}_{n-1}, \tilde{z}_n) = (\sqrt{-1}z_1, \dots, \sqrt{-1}z_{n-1}, -z_n)$  of  $\mathbb{C}^n$ , we may let

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & \sqrt{-1} \sin \beta & \cos \beta \\ \mathbf{0}_{(n-1) \times 1}^t & -\cos \beta & -\sqrt{-1} \sin \beta \end{bmatrix}.$$

Finally applying the automorphism  $(\tilde{w}_1, \dots, \tilde{w}_{n-1}, \tilde{w}_{n+1}) = (w_1, \dots, w_{n-1}, -w_{n+1})$  of  $D_{n+1}^{IV}$ ,

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 1} & \mathbf{0}_{(n-1) \times 1} \\ \mathbf{0}_{(n-1) \times 1}^t & \sqrt{-1} \sin \beta & \cos \beta \\ \mathbf{0}_{(n-1) \times 1}^t & \cos \beta & \sqrt{-1} \sin \beta \end{bmatrix}.$$

This is the matrix in (28). By solving the system

$$\left( z_1, \dots, z_n, \frac{1}{2} \sum_{j=1}^{n+1} f_j^2(z) \right) = (f_1(z), \dots, f_{n+1}(z)) \cdot \mathbf{U},$$

we obtain that  $F$  is equivalent to  $I_{n,\beta}$  in (13) for some  $\beta \in [0, \pi/4)$ . This establishes Theorem 4.2.  $\square$

**Theorem 4.3.** *For any  $\theta \in [0, \pi/4)$ ,  $I_{n,\theta} : \mathbb{B}^n \rightarrow D_{n+1}^{IV}$  given in (13) is equivalent to  $I_{n,0}$ .*

*Proof.* We first apply the Borel embedding to embed  $\mathbb{B}^n$  as an open subset of  $\mathbb{P}^n$  and  $D_{n+1}^{IV}$  as an open subset of  $\mathbb{Q}^{n+1} \subset \mathbb{P}^{n+2}$ , where the Borel embedding is given by

$$z = (z_1, \dots, z_{n+1}) \in D_{n+1}^{IV} \rightarrow \left[ z_1, \dots, z_{n+1}, \frac{1 + \frac{1}{2}zz^t}{\sqrt{2}}, \frac{(1 - \frac{1}{2}zz^t)}{\sqrt{-2}} \right] \in \mathbb{Q}^{n+1} \subset \mathbb{P}^{n+2}.$$

We write  $[z, s] = [z_1, \dots, z_n, s]$  to denote the homogeneous coordinates in  $\mathbb{P}^n$ . Then under homogeneous coordinates,  $I_{n,\theta}$  is identified with

$$\mathcal{I}_{n,\theta}(z, s) = [z_1, \dots, z_{n-1}, \phi_{n,\theta}(z, s), \phi_{n+1,\theta}(z, s), \phi_{n+2,\theta}(z, s), \phi_{n+3,\theta}(z, s)] \quad (29)$$

from  $\mathbb{P}^n$  to  $\mathbb{P}^{n+2}$  where

$$\begin{aligned} \phi_{n,\theta}(z, s) &= \frac{(\cos \theta s + \sqrt{-1} \sin \theta z_n) - \cos \theta \sqrt{H_\theta(z, s)}}{\cos(2\theta)}; \\ \phi_{n+1,\theta}(z, s) &= \frac{(-\sqrt{-1} \sin \theta s + \cos \theta z_n) + \sqrt{-1} \sin \theta \sqrt{H_\theta(z, s)}}{\cos(2\theta)}; \\ \phi_{n+2,\theta}(z, s) &= \frac{1 + \cos(2\theta)}{\sqrt{2} \cos(2\theta)} s + \frac{\sqrt{-1} \tan(2\theta) z_n}{\sqrt{2}} - \frac{1}{\sqrt{2} \cos(2\theta)} \sqrt{H_\theta(z, s)}; \\ \phi_{n+3,\theta}(z, s) &= \frac{\cos(2\theta) - 1}{\sqrt{-2} \cos(2\theta)} s - \frac{\sqrt{-1} \tan(2\theta) z_n}{\sqrt{2}} + \frac{1}{\sqrt{2} \cos(2\theta)} \sqrt{H_\theta(z, s)}; \\ H_\theta(z, s) &= s^2 + 2\sqrt{-1} \sin(2\theta) z_n s - z_n^2 - \cos(2\theta) \sum_{j=1}^{n-1} z_j^2. \end{aligned} \quad (30)$$

In particular,

$$\mathcal{I}_{n,0}(z, s) = \left[ z_1, \dots, z_{n-1}, s - \sqrt{H_0}, z_n, \frac{2s - \sqrt{H_0}}{\sqrt{2}}, \frac{\sqrt{H_0}}{\sqrt{-2}} \right].$$

Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\cos \theta}{\sqrt{\cos(2\theta)}} & \frac{-\sin \theta}{\sqrt{\cos(2\theta)}} \sqrt{-1} \\ \mathbf{0} & \frac{\sin \theta}{\sqrt{\cos(2\theta)}} \sqrt{-1} & \frac{\cos \theta}{\sqrt{\cos(2\theta)}} \end{bmatrix} \in U(n, 1) = \text{Aut}(\mathbb{B}^n),$$

and define  $\hat{\mathcal{I}}_{n,\theta}(z, s) = \mathcal{I}_{n,\theta}((z, s) \cdot \mathbf{B})$ . Then it follows from the straightforward calculation that

$$\hat{\mathcal{I}}_{n,\theta} = \left[ z_1, \dots, z_{n-1}, \frac{s - \cos \theta \sqrt{H_0}}{\sqrt{\cos(2\theta)}}, \frac{z_n + \sin \theta \sqrt{-H_0}}{\sqrt{\cos(2\theta)}}, \frac{2 \cos \theta s - \sqrt{H_0}}{\sqrt{2 \cos(2\theta)}}, \frac{-2 \sin \theta z_n - \sqrt{-H_0}}{\sqrt{2 \cos(2\theta)}} \right].$$

Let  $T = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{(n-1) \times 4} \\ \mathbf{0}_{4 \times (n-1)} & \mathbf{V} \end{bmatrix}$  with

$$\mathbf{V} = \frac{1}{\sqrt{\cos(2\theta)}} \begin{bmatrix} 1 - 4 \sin^2(\theta/2) & 0 & 2\sqrt{2} \sin^2(\theta/2) & 0 \\ 0 & 1 & 0 & -\sqrt{2} \sin \theta \\ 2\sqrt{2} \sin^2(\theta/2) & 0 & 1 - 4 \sin^2(\theta/2) & 0 \\ 0 & -\sqrt{2} \sin \theta & 0 & 1 \end{bmatrix}.$$

Then one can verify that  $T \in \text{Aut}(D_{n+1}^{IV})$  and

$$\begin{aligned} & \mathcal{I}_{n,0} \cdot T \\ &= \left( z_1, \dots, z_{n-1}, s - \sqrt{H_0}, z_n, \frac{2s - \sqrt{H_0}}{\sqrt{2}}, \frac{\sqrt{H_0}}{\sqrt{-2}} \right) \cdot T \\ &= \left( z_1, \dots, z_{n-1}, \frac{s - \cos \theta \sqrt{H_0}}{\sqrt{\cos(2\theta)}}, \frac{z_n + \sin \theta \sqrt{-H_0}}{\sqrt{\cos(2\theta)}}, \frac{2 \cos \theta s - \sqrt{H_0}}{\sqrt{2 \cos(2\theta)}}, \frac{-2 \sin \theta z_n - \sqrt{-H_0}}{\sqrt{2 \cos(2\theta)}} \right) \\ &= \hat{\mathcal{I}}_{n,\theta} = \mathcal{I}_{n,\theta}((z, s) \cdot \mathbf{B}). \end{aligned}$$

This implies that  $I_{n,\theta}$  is equivalent to  $I_{n,0}$ . □

Note that a rational map from  $\mathbb{B}^n \rightarrow D_{n+1}^{IV}$  cannot be equivalent to an irrational map. Thus combining Theorem 4.2, 4.3, we obtain the classification result Theorem 1.4 for holomorphic isometries from  $\mathbb{B}^n$  into  $D_{n+1}^{IV}$ .

**Remark 4.4.**  $n = 1$  is a special case ( $n + 1 = 2n$ ). Let  $F$  be a holomorphic isometry from  $\Delta \subset \mathbb{C}$  to  $D_2^{IV}$  satisfying  $F^* \omega_{D_2^{IV}} = \omega_\Delta$ . It follows from the same argument of Theorem 1.4 that  $F$  is either equivalent to the totally geodesic embedding  $(\sqrt{2}z/2, \sqrt{-2}z/2)$  or equivalent to the non-totally geodesic embedding  $I_{1,0}$ . Note that  $D_2^{IV}$  is biholomorphic to bidisc  $\Delta^2 \subset \mathbb{C}^2$ . In the Euclidean coordinate of  $\Delta^2$ , the first map is  $z \rightarrow (z, 0)$  and  $I_{1,0}$  is the square root embedding constructed by Mok [M4]. This classification result from  $\Delta$  to  $\Delta^2$  was obtained earlier by Ng (cf. [Ng1] Theorem 7.1).

## Appendix

Theorem 3.2 can be proved using the idea from [BEH2] in a very similar manner. We first introduce the following notations (See more details in [BH] and [BEH2]). Assume  $N \geq 3$  and let  $\mathbb{H}_1^N \subset \mathbb{C}^N$  be the hyperquadric of signature 1 defined by

$$\mathbb{H}_1^N = \{(z, w = u + \sqrt{-1}v) \in \mathbb{C}^N \mid v = \sum_{j=1}^{N-2} |z_j|^2 - |z_{N-1}|^2\},$$

where  $z = (z_1, \dots, z_{N-1})$ . Let  $\mathbb{H}^n$  be the Heisenberg hypersurface

$$\mathbb{H}^n = \{(z, w = u + \sqrt{-1}v) \in \mathbb{C}^n \mid v = \sum_{j=1}^{n-1} |z_j|^2\},$$

where  $z = (z_1, \dots, z_{n-1})$ . To prove Theorem 3.2, it suffices to show the following theorem.

**Theorem 4.5.** *Assume  $n \geq 4, n+1 \leq N \leq 2n-2$  and let  $F$  be a holomorphic map from a connected open set  $U \subset \mathbb{C}^n$  containing 0 to  $\mathbb{C}^N$ . If  $F(U \cap \mathbb{H}^n) \subset \mathbb{H}_1^N$  and  $F(U) \not\subset \mathbb{H}_1^N$ , then*

- *$F$  is equivalent to  $(z_1, \dots, z_{n-1}, 0, w)$  if  $N = n+1$ ;*
- *$F$  is equivalent to  $(z_1, \dots, z_n, \mathbf{0}, \psi, \psi, w)$  if  $N > n+1$ , where  $\psi$  is some holomorphic function on  $U$ .*

To prove Theorem 4.5, we will first establish the following proposition:

**Proposition 4.6.** *Let  $F = (\tilde{f}, g)$  be a holomorphic map from an open connected set  $U \subset \mathbb{C}^n$  containing 0 to  $\mathbb{C}^N$  with  $n \geq 4, n+1 \leq N \leq 2n-2$ . Assume that  $F(U \cap \mathbb{H}^n) \subset \mathbb{H}_1^N$ ,  $F(0) = 0$  and  $F$  is CR transversal at 0. Then  $\frac{\partial g}{\partial w}(0) > 0$  and  $F$  is*

- *equivalent to  $(z_1, \dots, z_{n-1}, 0, w)$  if  $N = n+1$ ;*
- *equivalent to  $(z_1, \dots, z_n, \mathbf{0}, \psi, \psi, w)$  if  $N > n+1$ , where  $\psi$  is some holomorphic function on  $U$ .*

We merely prove the case  $N > n+1$  in Proposition 4.6 and the other case follows similarly.

**Proof of Proposition 4.6.** We first show  $\frac{\partial g}{\partial w}(0) > 0$ . Write

$$F = (\tilde{f}, g) = (f, \varphi, g) = (f_1, \dots, f_{n-1}, \varphi_1, \dots, \varphi_{N-n}, g)$$

and write the tangent vector fields of  $\mathbb{H}^n$  at 0

$$\begin{aligned} L_i &= \frac{\sqrt{-1}}{2} \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial w}, 1 \leq i \leq n-1, \\ T &= \frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}. \end{aligned}$$

$F(U \cap \mathbb{H}^n) \subset \mathbb{H}_1^N$  implies that

$$\rho(F, \bar{F}) = -\frac{g - \bar{g}}{2\sqrt{-1}} + |f|^2 + \sum_{i=1}^{N-n-1} |\varphi_i|^2 - |\varphi_{N-n}|^2 = 0 \quad (31)$$

holds on  $\mathbb{H}^n$ . Here  $|f|^2 = \sum_{i=1}^{n-1} |f_i|^2$ . Applying  $L_i^k$  to (31) and evaluating at 0, we get

$$\frac{\partial^k g}{\partial z_i^k}(0) = 0.$$

Applying  $T$  to (31) and evaluating at 0, we get

$$\frac{\partial g}{\partial w}(0) = \overline{\frac{\partial g}{\partial w}(0)}.$$

This implies  $\frac{\partial g}{\partial w}(0) = \lambda \in \mathbb{R}$ . Since  $F$  is transversal at 0, we have  $\lambda \neq 0$ . We now write

$$\begin{aligned} g &= \lambda w + O(|(z, w)|^2), \\ \tilde{f}_j &= b_j w + \sum_{i=1}^n a_{ij} z_i + O(|(z, w)|^2), \end{aligned}$$

where  $b_j \in \mathbb{C}, a_{ij} \in \mathbb{C}$ . The second equation can be rewritten as

$$\tilde{f} = w(b_1, \dots, b_{N-1}) + (z_1, \dots, z_{n-1})A + \hat{f},$$

where  $A = (a_{ij})_{(n-1) \times (N-1)}$  is an  $(n-1) \times (N-1)$  matrix and  $\hat{f} = O(|(z, w)|^2)$ . We call a function  $h$  on  $U \cap \mathbb{H}^n$  has weighted degree  $s$ , denoted by  $h \in o_{wt}(s)$ , if

$$\lim_{t \rightarrow 0^+} \frac{h(tz, t\bar{z}, t^2 u)}{|t|^s} = 0.$$

For a smooth function  $h(z, \bar{z}, u)$  defined in  $U \cap \mathbb{H}^n$ , we denote by  $h^{(k)}(z, \bar{z}, u)$  the sum of terms of weighted degree  $k$  in the Taylor expansion of  $h$  at 0. We also denote by  $h^{(k)}(z, \bar{z}, u)$  a weighted homogeneous polynomial of degree  $k$ . When  $h^{(k)}(z, \bar{z}, u)$  extends to a weighted holomorphic polynomial of degree  $k$ , we write it as  $h^{(k)}(z, w)$  or  $h^{(k)}(z)$  if it depends only on  $z$ .

Note that (31) implies that

$$\frac{g - \bar{g}}{2\sqrt{-1}} = |f|^2 + \sum_{i=1}^{N-n-1} |\varphi_i|^2 - |\varphi_{N-n}|^2 \quad (32)$$

holds on  $w = u + \sqrt{-1}|z|^2$  near 0. We collect terms of weighted degree two on both sides of (32) to get

$$\lambda|z|^2 = zAE(N-2, N-1)\bar{A}^t \bar{z}^t, \quad (33)$$

which further implies

$$\lambda I_n = AE(N-2, N-1)\bar{A}^t.$$

Here  $E(N-2, N-1)$  denotes the  $(N-1) \times (N-1)$  diagonal matrix with its first  $(N-2)$  diagonal element  $+1$  and the rest  $-1$ . It follows from linear algebra that  $\lambda > 0$ .

We now fix some notations. For two  $m$ -tuples  $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m)$  of complex numbers, we write  $\langle x, y \rangle = \sum_{j=1}^m x_j y_j$  and  $|x|^2 = \langle x, \bar{x} \rangle$  and write  $\langle x, y \rangle_1 = \sum_{j=1}^{m-1} x_j y_j - x_m y_m$  and  $|x|_1^2 = \langle x, \bar{x} \rangle_1$ .

In this following context, to make notations easier, we will assume  $N = 2n - 2$ . The proof of other cases is of no significant difference. By the same argument as in [BH] (cf. Lemma 2.2), we can assume that  $F$  has the following normalization

$$\begin{aligned} f(z, w) &= z + \frac{\sqrt{-1}}{2} a^{(1)}(z)w + o_{wt}(3), \\ \varphi(z, w) &= \varphi^{(2)}(z) + o_{wt}(2), \\ g(z, w) &= w + o_{wt}(4), \end{aligned} \quad (34)$$

with

$$\langle a^{(1)}(z), \bar{z} \rangle |z|^2 = \langle \varphi^{(2)}(z), \overline{\varphi^{(2)}(z)} \rangle_1.$$

By Huang's lemma (cf. [Hu1] Lemma 3.2 or [BEH2] Lemma 2.1),

$$a^{(1)}(z) \equiv 0 \text{ and } \langle \varphi^{(2)}(z), \overline{\varphi^{(2)}(z)} \rangle_1 \equiv 0.$$

Therefore,

$$f^{(3)} \equiv 0.$$

Now suppose that we have obtained

$$g^{(t)} \equiv 0 \text{ and } f^{(t-1)} \equiv 0 \quad (35)$$

for  $3 \leq t < s$ . (Note that (35) is already proved for  $s = 5$ .) Now by collecting terms of weighted degree  $s$ , we obtain

$$\text{Im}\{g^{(s)}(z, w) - 2\sqrt{-1} \langle \bar{z}, f^{(s-1)}(z, w) \rangle\} = \sum_{s_1+s_2=s} \langle \varphi^{(s_1)}(z, w), \overline{\varphi^{(s_1)}(z, w)} \rangle_1 \quad (36)$$

when  $w = u + \sqrt{-1} \langle z, \bar{z} \rangle$ . We shall use the notation

$$\mathcal{L}(p, q)(z, \bar{z}, u) = \text{Im}\{q(z, w) - 2\sqrt{-1} \langle \bar{z}, p(z, w) \rangle\}|_{w=u+\sqrt{-1}\langle z, \bar{z} \rangle},$$

where  $p(z, w) = (p_1(z, w), \dots, p_{n-1}(z, w))$  and  $q(z, w)$  are holomorphic polynomials. Equation (36) can be written as

$$\mathcal{L}(f^{(s-1)}, g^{(s)})(z, \bar{z}, u) = \sum_{s_1+s_2=s} \langle \varphi^{(s_1)}(z, w), \overline{\varphi^{(s_2)}(z, w)} \rangle_1 |_{w=u+\sqrt{-1}\langle z, \bar{z} \rangle}. \quad (37)$$

The following result is crucial in the proof of Theorem 1.2.

**Lemma 4.7.** *Let  $F = (f, \varphi, g)$  be any normalized map as above sending an open piece  $M$  of  $\mathbb{H}^n$  near 0 into  $\mathbb{H}_1^N$  with  $N \leq 2n - 2$ . Assume that for all  $3 \leq t \leq 2(s^* - 1)$ ,*

$$f^{(t-1)} \equiv 0, g^{(t)} \equiv 0, \langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}} \rangle_1 \equiv 0 \quad (38)$$

for any  $(s_1, s_2)$  with  $s_1 + s_2 = t$ . Then (38) holds also for  $t = 2(s^* - 1) + 1$  and  $t = 2s^*$  for any such a map  $F$ .

Once Lemma 4.7 has been proved, we conclude by induction that (38) holds for all  $t \geq 3$ . We shall need some notation and results from [BEH2] and [EHZ2]. Given a real-valued power series  $A(z, \bar{z}, w, \bar{w})$ , we use the expansion

$$A(z, \bar{z}, w, \bar{w}) = \sum_{\mu\nu\gamma\delta} A_{\mu\nu\gamma\delta}(z, \bar{z}) w^\gamma \bar{w}^\delta$$

for  $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$  where  $A_{\mu\nu\gamma\delta}(z, \bar{z})$  is a bihomogeneous polynomial in  $(z, \bar{z})$  of bidegree  $(\mu, \nu)$  for every  $(\mu, \nu, \gamma, \delta) \in \mathbb{Z}_+^4$ . We recall from [EHZ2] that  $A(z, \bar{z}, w, \bar{w})$  belongs to the class  $\tilde{S}_k$  for a positive integer  $k$  if  $A$  vanishes at least up to order 2 at 0 and for every  $(\mu, \nu, \gamma, \delta) \in \mathbb{Z}_+^4$ , we have

$$A_{\mu\nu\gamma\delta}(z, \bar{z}) w^\gamma \bar{w}^\delta = \sum_{j=1}^k p_j(z, w) \overline{q_j(z, w)},$$

where  $p_j, q_j$  are homogeneous holomorphic polynomials of the appropriate degree. We recall the following result from [EHZ2].

**Theorem 4.8.** [EHZ2] *Let  $A(z, \bar{z}, w, \bar{w})$  be a real-valued weighted homogeneous polynomial of degree  $s \geq 5$  and assume that  $A \in \tilde{S}_{n-2}$ . If  $p(z, w) = (p_1(z, w), \dots, p_{n-1}(z, w))$  and  $q(z, w)$  are weighted homogeneous holomorphic polynomials of degree  $s - 1$  and  $s$  respectively, such that*

$$\mathcal{L}(p, q)(z, \bar{z}, u) = A(z, \bar{z}, w, \bar{w})|_{w=u+\sqrt{-1}|z|^2}.$$

Then

$$p \equiv 0, q \equiv 0 \text{ and } A \equiv 0.$$



**Proof of Lemma 4.7.** As in [BEH2], we shall first prove that (38) holds for  $t = 2(s^* - 1) + 1 = 2s^* - 1$  with  $s^* \geq 3$ . Recall the hypotheses in Lemma 4.7 implies that (37) holds with  $s = t$ . Note that the hypotheses also imply that  $\langle \varphi^{(s_1)}, \overline{\varphi^{(s_1)}} \rangle_1 \equiv 0$  for  $2 \leq s_1 \leq s^* - 1$ . By a lemma of D'Angelo [D2], we conclude that there are constants  $a_j^{s_1}$  such that

$$\varphi_j^{(s_1)} = a_j^{s_1} \varphi_{n-2}^{(s_1)}, \text{ for } 1 \leq j \leq n-3, 2 \leq s_1 \leq s^* - 1. \quad (39)$$

Now if  $s_1 + s_2 = 2s^* - 1$ , then  $\min(s_1, s_2) \leq s^* - 1$ . Without loss of generality, assume  $s_1 = \min(s_1, s_2)$ . Then it follows from (39) that

$$\langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}} \rangle_1 = \varphi_{n-2}^{(s_1)} \overline{q^{(s_2)}},$$

where  $q^{(s_2)}(z, w) = \sum_{j=1}^{n-3} a_j^{s_1} \overline{\varphi_j^{(s_2)}} - \overline{\varphi_{n-2}^{(s_2)}}$ . Since  $1 \leq n-2$ , it follows that  $A(z, \bar{z}, w, \bar{w}) := \sum_{s_1+s_2=2s^*-1} \langle \varphi^{(s_1)}(z, w), \overline{\varphi^{(s_2)}(z, w)} \rangle_1$  belongs to  $\tilde{S}_{n-2}$ . It follows from (37) and Theorem 4.8 that  $\overline{f^{(s-1)}} \equiv 0, g^{(s)} \equiv 0$  where  $s = 2s^* - 1$  and  $A \equiv 0$ . By the definition of  $A$ , we have  $\langle \varphi^{(s_1)}, \overline{\varphi^{(s_2)}} \rangle_1 \equiv 0$  for  $s_1 + s_2 = 2s^* - 1$ .

It remains to show (38) for  $t = 2s^* \geq 6$ . We first complexify (37) with  $s = 2s^*$ :

$$\begin{aligned} g^{(2s^*)}(z, w) - \overline{g^{(2s^*)}(\xi, \eta)} - 2\sqrt{-1} \langle \bar{\xi}, f^{(2s^*-1)}(z, w) \rangle - 2\sqrt{-1} \langle z, \overline{f^{(2s^*-1)}(\xi, \eta)} \rangle \\ = 2\sqrt{-1} \sum_k \langle \varphi^{(k)}(z, w), \overline{\varphi^{(2s^*-k)}(\xi, \eta)} \rangle_1. \end{aligned} \quad (40)$$

Write

$$L_j = \frac{\partial}{\partial z_j} + 2\sqrt{-1} \bar{\xi}_j \frac{\partial}{\partial w}, \quad 1 \leq j \leq n-1$$

be the tangent vector field of the complex hypersurface defined by  $w = \bar{\eta} + 2\sqrt{-1} \langle z, \bar{\xi} \rangle$ . We will first prove

$$f^{(2s^*-1)}(z, w) \equiv 0, g^{(2s^*)} \equiv 0. \quad (41)$$

We begin by establishing the following proposition.

**Proposition 4.9.** *For  $1 \leq j \leq n-1$ , we have*

$$f_j^{2s^*-1}(z, w) = a_j^{(1)}(z) w^{s^*-1}, g^{(2s^*)}(z, w) = d^{(0)} w^{s^*} \text{ and } a^{(1)}(z) = d^{(0)} z_j.$$

**Proof of Proposition.** Applying  $L_j$  to (40), we have on  $w = \bar{\eta} + 2\sqrt{-1} \langle z, \bar{\xi} \rangle$ ,

$$\begin{aligned} L_j (g^{(2s^*)}(z, w)) - 2\sqrt{-1} \langle \bar{\xi}, L_j f^{(2s^*-1)}(z, w) \rangle - 2\sqrt{-1} \overline{f^{(2s^*-1)}(\xi, \eta)} \\ = 2\sqrt{-1} \sum_k \langle L_j \varphi^{(k)}(z, w), \overline{\varphi^{(2s^*-k)}(\xi, \eta)} \rangle_1. \end{aligned}$$

Write

$$\begin{aligned} f_j^{(s-1)}(z, w) &= \sum a_j^{(\tau_j)}(z) w^{\tau_j}; \\ \varphi^{(k)} &= \sum b_k^{(\mu_k)}(z) w^{(\mu_k^*)}; \\ g^{(s)}(z, w) &= \sum d^{(j)}(z) w^{n_s^j}, \end{aligned}$$

where sums run over all indices such that  $\tau_j + 2\tau_s^j = s - 1$ ,  $\mu_k + 2\mu_k^* = k$ ,  $j + 2n_s^j = s$  with  $s = 2s^*$ . Letting  $w = 0$ ,  $\eta = 2\sqrt{-1} < \bar{z}, \xi >$  and collecting terms of degree  $k > 2$  in  $\xi$  and degree  $P$  in  $z$ . We obtain

$$\begin{aligned} & 2\sqrt{-1} \overline{a_j^{(K-P)} \eta^P} \\ &= 2\sqrt{-1} \sum_{k=2}^{s-2} < \varphi_{z_j}^{(k)}(z, 0), \overline{\sum b_{s-k}^{(\mu_{s-k})}(\xi) \eta^{\mu_{s-k}^*}} >_1 - 4 \sum_{k'=3}^{s-2} \bar{\xi}_j < \varphi_w^{(k')}(z, 0), \overline{\sum b_{s-k'}^{(\mu_{s-k'})}(\xi) \eta^{\mu_{s-k'}^*}} >_1, \end{aligned} \quad (42)$$

where the sum inside  $< \cdot, \cdot >_1$  run over the indices  $\mu_{s-k} + \mu_{s-k}^* = K$ ,  $k - 1 + \mu_{s-k}^* = P$  and  $\mu_{s-k'} + \mu_{s-k'}^* + 1 = K$ ,  $k' - 2 + \mu_{s-k'}^* = P$ . Note that the solution only exist when  $K + P + 1 = s = 2s^*$ . Letting  $K = s^* + p$ ,  $P = s^* - p - 1$ , we then get  $\mu_{s-k} = k + 2p$ ,  $\mu_{s-k}^* = s^* - p - k$  and  $\mu_{s-k'} = k' + 2p - 2$ ,  $\mu_{s-k'}^* = s^* - p - k' + 1$ . Since both  $\mu_{s-k}^*$  and  $\mu_{s-k'}^*$  must be nonnegative, one has  $2 \leq k \leq s^* - p$ ,  $3 \leq k' \leq s^* - p + 1$ . We now rewrite (42) as

$$\begin{aligned} & - 2\sqrt{-1} \overline{a_j^{(2p+1)}(\xi) \eta^{s^*-p-1}} \\ &= 2\sqrt{-1} \sum_{k=2}^{s^*-p} < \varphi_{z_j}^{(k)}(z, 0), \overline{b_{s-k}^{(k+2p)}(\xi)} >_1 \overline{\eta^{s^*-p-k}} - 4 \sum_{k'=3}^{s^*-p+1} \bar{\xi}_j < \varphi_w^{(k')}(z, 0), \overline{b_{s-k'}^{(k'+2p-2)}(\xi)} >_1 \overline{\eta^{s^*+1-p-k'}} \end{aligned}$$

which can further be rewritten as

$$\begin{aligned} & - 2\sqrt{-1} \overline{a_j^{(2p+1)}(\xi) \eta^{s^*-p-1}} \\ &= \sum_{k=2}^{s^*-p} \left( 2\sqrt{-1} < \varphi_{z_j}^{(k)}(z, 0), \overline{b_{s-k}^{(k+2p)}(\xi)} >_1 - 4\bar{\xi}_j < \varphi_w^{(k+1)}(z, 0), \overline{b_{s-k-1}^{(k+2p-1)}(\xi)} >_1 \right) \overline{\eta^{s^*-p-k}}. \end{aligned} \quad (43)$$

This equation is valid for all  $p = 0, \dots, s^* - 1$ . If  $p = s^* - 1$ , the sum on the right hand side is void. Note that for any  $q \leq s^* - 1$ , we can use (39) to rewrite

$$\begin{aligned} < \varphi_{z_j}^{(q)}(z, 0), \overline{b_{s-q}^{(q+2p)}(\xi)} >_1 = \sum_{i=1}^{n-3} \varphi_{i, z_j}^{(q)}(z, 0) \overline{b_{i, s-q}^{(q+2p)}(\xi)} - \varphi_{n-2, z_j}^{(q)}(z, 0) \overline{b_{n-2, s_q}^{(q+2p)}(\xi)} \\ &= \varphi_{n-2, z_j}^{(q)}(z, 0) \overline{C(\xi)} \end{aligned}$$

with

$$C(\xi) = \sum_{i=1}^{n-3} \overline{a_j^s b_{i,s-q}^{(q+2p)}(\xi)} - \overline{b_{n-2,s-q}^{(q+2p)}(\xi)}.$$

We can make a similar substitution in  $\langle \varphi_w^{(q)}(z, 0), \overline{b_{s-k-1}^{(q+2p)}(\xi)} \rangle_1$  for any  $q \leq s^* - 1$ . Therefore, if  $2 \leq q \leq s^* - 1$ , we then conclude by Corollary 2.2 in [BEH2], as  $2 \leq n - 2$ , that

$$a_j^{(2p+1)}(z) \equiv 0, \text{ for } j = 1, \dots, n-1 \text{ and } p = 2, \dots, s^* - 1. \quad (44)$$

When  $p = 1$ , (43) can be written as

$$\begin{aligned} -2\sqrt{-1} \overline{a_j^{(3)}(\xi) \eta^{s^*}} &= \sum_{k=1}^{s^*-2} \left( 2\sqrt{-1} \langle \varphi_{z_j}^{(k)}(z, 0), \overline{b_{s-k}^{(k+2)}(\xi)} \rangle_1 - 4\bar{\xi}_j \langle \varphi_w^{(k+1)}(z, 0), \overline{b_{s-k-1}^{(k+1)}(\xi)} \rangle_1 \right) \overline{\eta^{s^*-1-k}} \\ &\quad + 2\sqrt{-1} \langle \varphi_{z_j}^{(s^*-1)}(z, 0), \overline{b_{s^*+1}^{(s^*+1)}(\xi)} \rangle_1 - 4\bar{\xi}_j \langle \varphi_w^{(s^*)}(z, 0), \overline{b_{s^*}^{(s^*)}(\xi)} \rangle_1 \end{aligned} \quad (45)$$

We now turn to the equation (40) in which we set  $w = 0$  and  $\eta = 2\sqrt{-1} \langle \bar{z}, \xi \rangle$ . Collecting terms of degree  $s^*$  in  $z$  and  $s^*$  in  $\xi$ , we obtain

$$-\overline{d^{(0)} \eta^{s^*}} - 2\sqrt{-1} \langle z, \overline{a^{(1)}(\xi)} \rangle \overline{\eta^{s^*-1}} = 2\sqrt{-1} \sum_{k=2}^{s^*} \langle \varphi^{(k)}(z, 0), \overline{b_{s-k}^{(k)}(\xi)} \rangle_1 \overline{\eta^{s^*-k}}.$$

This can be rewritten as

$$-\overline{d^{(0)} \eta^{s^*}} - 2\sqrt{-1} \langle z, \overline{a^{(1)}(\xi)} \rangle \overline{\eta^{s^*-1}} - 2\sqrt{-1} \sum_{k=2}^{s^*-1} \langle b_k^{(k)}(z), \overline{b_{s-k}^{(k)}(\xi)} \rangle \overline{\eta^{s^*-k}} = 2\sqrt{-1} \langle b_{s^*}^{(s^*)}(z), \overline{b_{s^*}^{(s^*)}(\xi)} \rangle_1.$$

Note the left hand side is divisible by  $\eta$  that is a summation of  $n-1$  terms, while  $\langle b_{s^*}^{(s^*)}(z), \overline{b_{s^*}^{(s^*)}(\xi)} \rangle_1$  is a summation of  $n-2$  terms, by Huang's lemma (cf. Lemma 3.2 in [Hu1] or Lemma 2.1 in [BEH2]), we have

$$\langle b_{s^*}^{(s^*)}(z), \overline{b_{s^*}^{(s^*)}(\xi)} \rangle_1 \equiv 0.$$

Therefore

$$b_{j s^*}^{s^*}(z) = A_j^{s^*} b_{(n-2)s^*}^{(s^*)}(z), \quad 1 \leq j \leq n-3$$

for some constant  $A_j^{s^*}$ . Here  $b_{j s^*}^{(s^*)}$  is the  $j$ -th component of the vector-valued function  $b_{s^*}^{(s^*)}(z)$ . Using Corollary 2.2 in [BEH2], we have

$$\langle b_k^{(k)}(z), \overline{b_{s-k}^{(k)}(\xi)} \rangle_1 \equiv 0 \text{ for } k = 2, \dots, s^* - 1 \quad (46)$$

and

$$\overline{d^{(0)} \eta} = 2\sqrt{-1} \langle z, \overline{a^{(1)}(\xi)} \rangle. \quad (47)$$

Equation (47) implies

$$a_j^{(1)}(z) = d^{(0)}z_j.$$

If we use (46) to substitute for  $b_{js^*}^{(s^*)}$  in (45) as we did before, then it follows from Corollary 2.2 in [BEH2] again that

$$a_j^{(3)} \equiv 0 \text{ for } j = 1, \dots, n-1. \quad (48)$$

(44) and (48) imply that

$$f^{2s^*-1}(z, w) = a^{(1)}(z)w^{s^*-1}.$$

To show  $g^{(2s^*)}(z, w) = d^{(0)}w^{s^*}$ , we go back to (40) in which we again set  $w = 0$  and  $\eta = -2\sqrt{-1} \langle \bar{z}, \xi \rangle$ . Note that we have proved that the degree of  $f^{2s^*-1}(\xi, \eta)$  in  $\xi$  is  $s^*$ . Thus, if we collect terms of degree  $K = s^* + p$  in  $\xi$  with  $p \geq 1$ , then we obtain

$$-\overline{d^{(2p)}(\xi)\eta^{s^*-p}} = 2\sqrt{-1} \sum_{k=2}^{s^*-p} \langle \varphi^{(k)}(z, 0), \sum \overline{b_{s-k}^{(\mu_{s-k})}(\xi)\eta^{\mu_{s-k}^*}} \rangle_1,$$

where the sum inside  $\langle \cdot, \cdot \rangle_1$  runs over  $\mu_{s-k} + \mu_{s-k}^* = s^* + p$ ,  $\mu_{s-k} + 2\mu_{s-k}^* = s - k$ . As before, the equation above can be rewritten as

$$-\overline{d^{(2p)}(\xi)\eta^{s^*-p}} = 2\sqrt{-1} \sum_{k=2}^{s^*-p} \langle \varphi^{(k)}(z, 0), \overline{b_{s-k}^{(k+2p)}(\xi)} \rangle \eta^{s^*-p-k} \text{ for } p = 1, \dots, s^*.$$

We then substitute for  $\varphi^{(k)}(z, 0)$  using (39) and apply Corollary 2.2 in [BEH2] to conclude

$$d^{(2p)}(\xi) \equiv 0 \text{ for } p = 1, \dots, s^*.$$

This establishes the proposition.

We will need the following lemma to complete the proof of (41).

**Lemma 4.10.** *Suppose that the hypotheses in Lemma 4.7 hold. Assume further that for any  $p \in M$ , we have*

$$\begin{aligned} f_p^{**}(z, w) &= z + a_p^{(1)}(z)w^{s^*-1} + O_{wt}(s) \\ g_p^{**}(z, w) &= w + d_p^{(0)}w^{s^*} + O_{wt}(s+1) \end{aligned}$$

where  $s = 2s^*$ . Then  $a_p^{(1)}(z) \equiv 0$ ,  $d_p^{(0)} = 0$  and hence

$$f(z, w) = z + O_{wt}(s), g(z, w) = w + O_{wt}(s+1).$$

*Proof.* One can apply the identical proof of Lemma 3.5 in [BEH2]. We need to employ the moving points trick that was first introduced in [Hu1] and has been extensively used in many literatures (cf. [Hu1], [BH], [BEH2]).  $\square$

To complete the induction step in the proof of Lemma 4.7, we must show that

$$\langle \varphi^{(k)}(z, w), \overline{\varphi^{(s-k)}(\xi, \eta)} \rangle_1 \equiv 0 \text{ for } k = 2, \dots, s-2$$

with  $s = 2s^*$ . This can be established by a very similar proof as [BEH2] (cf. pp 1649-1655) using the same idea as above with the following difference. In our setting, one should apply Huang's lemma (cf. Lemma 3.2 [Hu1] or Lemma 2.1 [BEH2] ) while [BEH2] uses Lemma 2.3. Here note Huang's lemma can be applied as by our assumption  $\langle \varphi, \bar{\varphi} \rangle_1$  is a summation with less terms than  $\langle z, \bar{\xi} \rangle$ , i.e.  $N - n < n - 1$ .  $\square$

Now we are able to finish the proof of Proposition 4.6. First we may assume that  $F$  satisfies the normalization (34). By Lemma 4.7, we conclude that

$$f(z, w) \equiv z, g(z, w) \equiv w, \langle \varphi(z, w), \overline{\varphi(\xi, \eta)} \rangle_1 \equiv 0.$$

By the result of D'Angelo [D2], there is a constant  $(n-3) \times (n-3)$  unitary matrix  $U$  such that

$$(\varphi_1, \dots, \varphi_{n-3}) \cdot U = (\varphi_{n-2}, 0, \dots, 0)$$

with  $(n-4)$  zero components on the right hand side of the equation. Letting  $\gamma$  be the automorphism of  $\mathbb{H}_1^N$  given by

$$\gamma(z, w) = (z_1, \dots, z_{n-1}, (z_n, \dots, z_{N-2}) \cdot U, z_{N-1}, w),$$

then  $\gamma \circ F$  satisfies the conclusion of Proposition 4.6. This finishes the proof of Proposition 4.6.

We now prove Theorem 4.5.

**Proof of Theorem 4.5.** We set

$$E = \{p \in \mathbb{H}^n \mid F \text{ is not CR transversal to } \mathbb{H}_1^N \text{ at } p\}.$$

The set  $E$  is a real analytic subvariety near 0 in  $\mathbb{H}^n$ . If  $E$  contains an open neighborhood of 0, then it follows from Theorem 1.1 in [BER2] (cf. [BH] Lemma 4.1) that  $F(U) \subset \mathbb{H}_1^N$ . Now assume  $F(U) \not\subset \mathbb{H}_1^N$ . Then  $E$  is a proper real analytic subvariety of  $\mathbb{H}^n$ . Consequently, there exists  $p \in \mathbb{H}^n$  near 0 such that  $F$  is CR transversal at  $p$ . We recall from [BEH2] that  $F$  is CR transversal to  $\mathbb{H}_1^N$  at  $p \in M$  if and only if  $\frac{\partial g_p}{\partial w}(0) \neq 0$ . Here  $F_p = (f_p, \varphi_p, g_p)$  (refer the definition of  $F_p$  in [BH]). Thus replacing  $F$  by  $F_p$  if necessary, we can assume  $p = 0$ . Then the result follows easily from Proposition 4.6.

Recall the well-known fact that the Cayley transformation biholomorphically maps  $\mathbb{H}_1^N$  into  $\partial\mathbb{B}_1^N$  minus a proper subvariety. Then Theorem 3.2 is a consequence of Theorem 4.5.

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