Holomorphic isometries from the unit ball into the irreducible classical bounded symmetric domain

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Abstract

We classify the holomorphic isometries from $\mathbb{B}^n$ into the classical domain D^{IV}_{n+1} of Type IV. More generally, we further study holomorphic isometries from $\mathbb{B}^n$ into D^{IV}_m for $m \geq n+1$.

1 Introduction

The study of local holomorphic isometries between bounded symmetric domains is motivated by the work of of Clozel-Ullmo [CU] in arithmetic geometry on commutators of modular correspondences on quotients $S := \Omega/\Gamma$ of an irreducible bounded symmetric domain Ω by torsion-free lattices Γ of the holomorphic automorphism group $\mathrm{Aut}(\Omega)$. Clozel-Ullmo characterized the modular correspondence among algebraic correspondence $Y \subset S \times S$ by assuming that the algebraic correspondence preserves the invariant metric or the invariant measure on Ω [CU]. Here $X \subset S \times S$ is a modular correspondence if X is a totally geodesic complex submanifold which descends from the graph of a holomorphic automorphism element of Ω. Their approach is to reduce this arithmetic geometric problem to the problem of local holomorphic isometries or holomorphic measure-preserving maps from Ω into the product of Ω (cf. [M3] [MN2] [Yu] for related problems).

More recently, Mok systematically studied this type of problem in the very general setting. Various extension and rigidity theorems are proved. Assume $n \geq 2$ and let $D \subset \mathbb{C}^n, \Omega \subset \mathbb{C}^N$ be bounded symmetric domain and D be irreducible. Assume $U \subset D$ is a connected open set and

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assume $F : U \to \Omega$ be a holomorphic isometry with respect to the (normalized) Bergman metric in the sense that $\lambda\omega_D = F^*\omega_\Omega$ holds on U, where ω_D, ω_Ω denote the (normalized) Bergman metric on D, Ω respectively. It is proved by Mok that F is an algebraic map in the sense that the graph of F is an affine subvariety in $\mathbb{C}^{n+N}$ and F extends to a proper holomorphic map from D into Ω [M4]. Moreover, Mok proved that F is totally geodesic if $\text{rank}(D) \geq 2$ [M4]. When $\text{rank}(D) = 1$, i.e. D is the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, and Ω is the product of unit balls, F is also totally geodesic by works of Mok, Ng and Yuan-Zhang [M2] [Ng1] [YZ]. However, when $D = \mathbb{B}^n$ and Ω is a bounded symmetric domain other than the product of unit balls, the total geodesy fails dramatically. In fact, assuming Ω is irreducible and $\text{rank}(\Omega) \geq 2$, Mok constructed non-totally geodesic holomorphic isometry from $\mathbb{B}^n$ into Ω by using the theory of variety of minimal rational tangents [M6] (cf. Theorem 2.1 in section 2). To classify the non-totally geodesic holomorphic isometries is the first motivation of our study. In Section 4, we classify all holomorphic isometries from $\mathbb{B}^n$ into Type IV classical domain D^{IV}_{n+1}. In fact, there are only two maps: one rational map and the other irrational map.

Theorem 1.1. *Let F be any holomorphic isometry from $\mathbb{B}^n$ into D^{IV}_{n+1} satisfying*

$$F^*\omega_{D^{IV}_{n+1}} = \omega_{\mathbb{B}^n}.$$

Then F is either equivalent to R^{IV}_n in (6) or equivalent to $I_{n,0}$ in (8).

We also study the holomorphic isometries from $\mathbb{B}^n$ to a classical symmetric domains of type IV of higher dimensions. We explore some interesting phenomena of such mappings. More precisely,

Theorem 1.2. *For $n+1 \leq m \leq 2n+2$, there are minimal rational and irrational holomorphic ismetries from $\mathbb{B}^n$ to D^{IV}_m. For $m > 2n+2$, there are no minimal holomorphic isometries from $\mathbb{B}^n$ to D^{IV}_m.*

Theorem 1.3. *For each $m \geq n+1$, there is a real-parameter family of mutually nonequivalent holomorphic isometries from $\mathbb{B}^n$ to D^{IV}_m.*

On the other hand, the holomorphic isometries from a bounded symmetric domain in $\mathbb{C}$, i.e. the unit disc Δ, into a bounded symmetric domain possess different phenomena. The so called p-th root map constructed by Mok (cf. [M5]) from Δ into the p-disc Δ^p indicates that the rigidity or total geodesy fails (cf. [Ng2] [Ch] for related results). Later Mok [M3][M7] and Mok-Ng [MN1] studied more carefully the asymptotic behavior of the second fundamental form associated to the holomorphic isometry at the boundary $\partial\Delta$ and many deep results were obtained there. Since all the previous-constructed non-totally geodesic holomorphic isometries from Δ into a bounded symmetric domain Ω produces singularities along $\partial\Delta$, Mok raised the question in [M5] (see also [M3]) regarding the boundary singularity set, which is answered in Theorem 4.1. To understand the boundary singularities is the second motivation of our study. To this end, we write down, in Section 4, explicit polynomial non-totally geodesic holomorphic isometries from $\mathbb{B}^n$ into classical symmetric domains of Type I, II, III and IV.

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2 Preliminaries

The classical symmetric domains of Cartan are realized in the following four types (cf. [H2] [M1]). Assume $q \leq p$ and let $M(p,q;\mathbb{C})$ denote the space of $p \times q$ matrices with entries of complex numbers. The type I domain is defined as

$$D^I_{p,q} = \{Z \in M(p,q;\mathbb{C}) | I_q - \overline{Z}^t Z > 0\}.$$

In particular, when $q = 1$, the type I domain is the complex unit ball $\mathbb{B}^p = \{z = (z_1, \cdots, z_p) \in \mathbb{C}^p || z|^2 < 1\}$ in $\mathbb{C}^p$. The type II and type III domains are submanifolds of $D^I_{n,n}$ defined as

$$D^{II}_n = \{Z \in D^I_{n,n} | Z = -Z^t\}$$

and

$$D^{III}_n = \{Z \in D^I_{n,n} | Z = Z^t\}.$$

The type IV domain is defined as

$$D^{IV}_n = \{Z = (z_1, \cdots, z_n) \in \mathbb{C}^n | Z\overline{Z}^t < 2 \text{ and } 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 > 0\}.$$

Let Ω be an irreducible bounded symmetric domain. The Bergman kernel function $K_\Omega(Z, \bar{Z})$ is explicitly given by

$$\begin{aligned}
K_\Omega(Z,\bar{Z}) &= c_I \left(\det(I_q - \overline{Z}^t Z)\right)^{-(p+q)} \text{ when } \Omega = D^I_{p,q};\\
K_\Omega(Z,\bar{Z}) &= c_{II} \left(\det(I_n - \overline{Z}^t Z)\right)^{-(n-1)} \text{ when } \Omega = D^{II}_n;\\
K_\Omega(Z,\bar{Z}) &= c_{III} \left(\det(I_n - \overline{Z}^t Z)\right)^{-(n+1)} \text{ when } \Omega = D^{III}_n;\\
K_\Omega(Z,\bar{Z}) &= c_{IV} \left(1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2\right)^{-n} \text{ when } \Omega = D^{IV}_n
\end{aligned} \tag{1}$$

where c_* are positive constants depending on n and the type of Ω(cf. [H2] [M1]). The Bergman metric

$$\omega^B_\Omega(Z) := \sqrt{-1}\partial\bar{\partial}\log K_\Omega(Z, \bar{Z})$$

on Ω is Kähler-Einstein as the Bergman kernel function is invariant under the holomorphic automorphisms. Note that the standard linear embedding $L(Z) = Z$ from D^{II}_n or D^{III}_n into $D^I_{n,n}$ is a totally geodesically holomorphic isometric embedding with respect to Bergman metrics with isometric constant $\frac{2n}{n-1}$ or $\frac{2n}{n+1}$ respectively.

Let S be the Hermitian symmetric space of compact type dual to Ω and $\delta \in H^2(S,\mathbb{Z})$ be the positive generator. It is well-known that the first Chern class $c_1(S) = (p+q)\delta, 2(n-1)\delta, (n+1)\delta$ or $n\delta$ when the classical domain $\Omega = D^I_{p,q}, D^{II}_n, D^{III}_n$ or D^{IV}_n respectively.

In this following context, for any irreducible bounded symmetric domain Ω, we write ω_Ω as the canonical Kähler-Einstein metric on Ω normalized so that minimal disks on Ω are of constant Gaussian curvature -2. In [M6], Mok constructed nonstandard holomorphic isometries from the complex unit ball into the irreducible bounded symmetric domain Ω when $\mathrm{rank}(\Omega) \geq 2$ by using the theory of varieties of minimal rational tangents. More precisely, Mok proved

Theorem 2.1 (Mok)**.** *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 and S its compact dual. Suppose the first Chern class $c_1(S) = (n_\Omega+1)\delta$, where δ is the positive generator of $H^2(S,\mathbb{Z})$.*

(i) If $F : \mathbb{B}^m \to \Omega$ is a holomorphic isometry, i.e.

$$F^*\omega_\Omega = \omega_{\mathbb{B}^m}, \tag{2}$$

then $m \leq n_\Omega$.

(ii) There exists a nonstandard holomorphic isometric embedding $G : \mathbb{B}^{n_\Omega} \to \Omega$ with $G^*\omega_\Omega = \omega_{\mathbb{B}^{n_\Omega}}$.

Remark 2.2. *We refer the number n_Ω to the maximal dimension of the complex unit ball that admits a holomorphic isometric embedding into Ω.*

Remark 2.3. *Write $z = (z_1, ..., z_m)$ as the coordinates of $\mathbb{C}^m$. Assuming $F(0) = 0$, then (2) is equivalent to following functional equations*

$$\begin{aligned}
1-|z|^2 &= \det(I_q - \overline{F(z)}^t F(z)) \ \text{when } \Omega = D^I_{p,q};\\
(1-|z|^2)^2 &= \det(I_n - \overline{F(z)}^t F(z)) \ \text{when } \Omega = D^{II}_n;\\
1-|z|^2 &= \det(I_n - \overline{F(z)}^t F(z)) \ \text{when } \Omega = D^{III}_n;\\
1-|z|^2 &= 1 - F(z)\overline{F(z)}^t + \frac{1}{4}|F(z)F(z)^t|^2 \ \text{when } \Omega = D^{IV}_n.
\end{aligned} \tag{3}$$

Furthermore, when m is the maximal dimension n_Ω, F is also a holomorphic isometry from $\mathbb{B}^{n_\Omega}$ into Ω with respect to Bergman metrics.

Definition 2.4. *Two proper holomorphic maps $F_1, F_2 : D_1 \to D_2$ between domains D_1 and D_2 are equivalent if there exist $\phi \in Aut(D_1), \psi \in Aut(D_2)$ such that*

$$\psi \circ F_1 \circ \phi = F_2.$$

3 Holomorphic isometries from $\mathbb{B}^n$ into D_{n+1}^{IV}

In this section we will present the classification of holomorphic isometries from $\mathbb{B}^n$ into D_{n+1}^{IV}. We start with a result on the isometric constants of such maps.

3.1 On isometric constants

We first derive the following theorem on isometric constants for holomorphic isometries. See a related result in [M6](Theorem 3).

Theorem 3.1. *Assume $n \geq 2$. Let $F : \mathbb{B}^n \to D_m^{IV}$ be a holomorphic isometric embedding with respect to Bergman metrics of isometric constant $\lambda > 0$, i.e.*

$$F^*\omega_{D_m^{IV}} = \lambda\omega_{\mathbb{B}^n}. \tag{4}$$

Then $\lambda = 1$.

Proof. Without loss of generality, we may assume $F(0) = 0$ by composing the automorphisms of $\mathbb{B}^n$ and D_m^{IV}. By the standard reduction, (4) is equivalent to

$$\left(1 - |z|^2\right)^{\lambda(n+1)/m} = 1 - F(z)\overline{F(z)}^t + \frac{1}{4}|F(z)F(z)^t|^2. \tag{5}$$

Note that the signature of the left hand side of (5) is either $(1, s)$ or $(2, s)$ for some integer $s \geq 0$, meaning that it can be written as linear combination of 1 or 2 sum of squares minus s sum of square of linearly independent holomorphic functions over positive real numbers. It fact, it is of signature $(1, s)$ if and only if $F(z)F(z)^t \equiv 0$. Obviously, the left hand side is of finite rank if and only if $\lambda(n+1)/m \in \mathbb{N}$ (cf. [Um]). Write $u = \lambda(n+1)/m$. Assume $u \geq 2$. Applying binomial formula,

$$\left(t^2 - \sum_{j=1}^{m} |z_j|^2\right)^u = \sum_{k=0}^{u} (-1)^k \binom{u}{k} (t^2)^k \left(\sum_{j=1}^{n} |z_j|^2\right)^{u-k}.$$

Note that the monomials on the right hand side are linearly independent and thus the right hand side is of signature (r', s') with $r' \geq 3$ if $n \geq 2$. Therefore, the left hand side and right hand side of (5) have different signatures and this contradicts to Theorem 3.2 in [Um]. □

Remark 3.2. *Let $n \geq 2$ and assume that $F : \mathbb{B}^n \to D_m^{IV}$ be a holomorphic isometric embedding with respect to Bergman metrics up to an isometric constant such that $F(0) = 0$. Then we always have*

$$1 - |z|^2 = 1 - F(z)\overline{F(z)}^t + \frac{1}{4}|F(z)F(z)^t|^2.$$

Moreover, Theorem 3.1 fails if $n = 1$ because of $F(z) = \left(\sqrt{2}z, 0\right) : \Delta \to D_2^{IV}$.

Remark 3.3. *The same argument yields that if $F : \mathbb{B}^n \to \Omega$ is a holomorphic isometric embedding from a unit ball into a classical symmetric domain with $F^*\omega_\Omega = \lambda\omega_{\mathbb{B}^n}$ of isometric constant λ, then λ is a positive integer when $\Omega = D_{p,q}^I, D_m^{II}$ or D_m^{III} respectively.*

3.2 Classifications of holomorphic isometries from $\mathbb{B}^n$ into D_{n+1}^{IV}

This subsection is devoted to establish Theorem 1.1. For that we will first describe the holomorphic automorphism group action on D_n^{IV} in terms of the Borel embedding (cf. [H1] [M1]). The hyperquadric $\mathbb{Q}^n$, the compact dual of D_n^{IV} is defined by $\mathbb{Q}^n := \{[z_1, \cdots, z_{n+2}] \in \mathbb{P}^{n+1} | z_1^2 + \cdots + z_n^2 = z_{n+1}^2 + z_{n+2}^2\}$. The Borel embedding $D_n^{IV} \subset \mathbb{Q}^n \subset \mathbb{P}^{n+1}$ is given by

$$Z = (z_1, \cdots, z_n) \to \left[z_1, \cdots, z_n, \frac{1 + \frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}ZZ^t}{\sqrt{-2}} \right].$$

The holomorphic automorphism group of D_n^{IV} is given by

$$\mathrm{Aut}(D_n^{IV}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in O(n, 2, \mathbb{R}) | \det(D) > 0, \right\}$$

where $A \in M(n, n, \mathbb{R}), B \in M(n, 2, \mathbb{R}), C \in M(2, n, \mathbb{R}), D \in M(2, 2, \mathbb{R})$. The automorphism group action is given in the following explicit way. Let $Z = (z_1, \cdots, z_n) \in D_n^{IV}$ and $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Aut}(D_n^{IV})$. Write $Z' = \left(\frac{1+\frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1-\frac{1}{2}ZZ^t}{\sqrt{-2}} \right)$. Then the action of T on D_n^{IV} is given by

$$T(Z) = \frac{ZA + Z'C}{(ZB + Z'D) \left(1/\sqrt{2}, \sqrt{-1/2} \right)^t}.$$

Rephrasing in homogenous coordinates, if the holomorphic automorphism maps $Z = (z_1, \cdots, z_n) \in D_n^{IV}$ to $W = (w_1, \cdots, w_n) \in D_n^{IV}$, then there exists $T \in \mathrm{Aut}(D_n^{IV})$ such that

$$\left[w_1, \cdots, w_n, \frac{1 + \frac{1}{2}WW^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}WW^t}{\sqrt{-2}} \right] = \left[z_1, \cdots, z_n, \frac{1 + \frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}ZZ^t}{\sqrt{-2}} \right] \cdot T.$$

In other words, there exists nonzero $\lambda \in \mathbb{C}$, such that

$$\left(w_1, \cdots, w_n, \frac{1 + \frac{1}{2}WW^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}WW^t}{\sqrt{-2}} \right) = \lambda \left(z_1, \cdots, z_n, \frac{1 + \frac{1}{2}ZZ^t}{\sqrt{2}}, \frac{1 - \frac{1}{2}ZZ^t}{\sqrt{-2}} \right) \cdot T.$$

Note that the isotropy group K_0 at the origin is $K_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in O(n, 2, \mathbb{R}) | \det(D) = 1 \right\} \cong O(n, \mathbb{R}) \times SO(2, \mathbb{R})$.

Write $Z = (z_1, ..., z_n)$ be the coordinates in $\mathbb{C}^n$ for $n \geq 2$. Let $R_n^{IV} : \mathbb{B}^n \to D_{n+1}^{IV}$ be defined as

$$R_n^{IV} = (f_1, ..., f_{n-1}, f_n, f_{n+1}), \tag{6}$$

where $f_i = z_i, 1 \leq i \leq n-1, f_n = \frac{P_n}{Q}, f_{n+1} = \frac{P_{n+1}}{Q}$,

$$P_n = \frac{1}{2} \sum_{i=1}^{n-1} z_i^2 - z_n^2 + z_n, P_{n+1} = -\sqrt{-1} \left(\frac{1}{2} \sum_{i=1}^{n-1} z_i^2 + z_n^2 - z_n \right), Q = \sqrt{2}(1 - z_n).$$

It is easy to verify that

$$\sum_{i=1}^n f_i^2 = \frac{\sum_{i=1}^{n-1} z_i^2}{1-z_n}, \quad |f_n|^2+|f_{n+1}|^2 = \frac{1}{4}\frac{|\sum_i^{n-1} z_i^2|^2}{|1-z_n|^2}+|z_n|^2.$$

It then follows that R_n^{IV} is a holomorphic isometry from $\mathbb{B}^n$ to D_{n+1}^{IV}. In fact, we will show that R_n^{IV} is the unique rational holomorphic isometry from $\mathbb{B}^n$ to D_{n+1}^{IV} up to holomorphic automorphisms.

For any $\theta \in [0,\pi/4)$, define $h_\theta(z) = 1+2\sqrt{-1}\sin(2\theta)z_n - z_n^2 - \cos(2\theta)\left(\sum_{j=1}^{n-1} z_j^2\right)$. Let $I_{n,\theta} = (f_1,\cdots,f_{n+1})$ be

$$\begin{aligned} f_1(z) &= z_1, \cdots, f_{n-1}(z) = z_{n-1}; \\ f_n(z) &= \frac{(\cos\theta+\sqrt{-1}\sin\theta z_n) - \cos\theta\sqrt{h_\theta(z)}}{\cos(2\theta)}; \\ f_{n+1}(z) &= \frac{(-\sqrt{-1}\sin\theta+\cos\theta z_n)+\sqrt{-1}\sin\theta\sqrt{h_\theta(z)}}{\cos(2\theta)}. \end{aligned} \tag{7}$$

Then $I_{n,\theta}$ is a holomorphic isometry from $\mathbb{B}^n$ to D_{n+1}^{IV}. In particular, when $\theta=0$,

$$I_{n,0} = \left(z_1,\cdots,z_{n-1}, 1-\sqrt{1-\sum_{j=1}^n z_j^2}, z_n\right). \tag{8}$$

We will also show that $I_{n,0}$ is the unique irrational holomorphic isometry from $\mathbb{B}^n$ to D_{n+1}^{IV} up to holomorphic automorphisms.

We are now at the position to prove Theorem 1.1. We first prove a result on isotropy equivalence.

Theorem 3.4. *Let $F:\mathbb{B}^n\to D_{n+1}^{IV}$ be a holomorphic isometric embedding satisfying $F(0)=0$ and*

$$F^*\omega_{D_{n+1}^{IV}} = \omega_{\mathbb{B}^n}. \tag{9}$$

Then F is isotropically equivalent to either the map R_n^{IV} in (6) or the map $I_{n,\theta}$ in (7) for some $\theta\in[0,\pi/4)$.

Proof. **First normalization:** Write $F=(f_1,\cdots,f_{n+1})$. By the isometry assumption, we have

$$\left(z_1,\cdots,z_n,\frac{1}{2}\sum_{j=1}^m f_j^2(z)\right) = (f_1(z),\cdots,f_{n+1}(z))\cdot\mathbf{U} \tag{10}$$

for some unitary $(n+1)\times(n+1)$ matrix $\mathbf{U}$ as the equation (43). Write $\mathbf{U}=(\mathbf{u_1},\cdots,\mathbf{u_{n+1}})$, where each $\mathbf{u_i}$ for $1\le i\le n+1$, is a column vector in $\mathbb{C}^{n+1}$. Write the first n-columns of U as $\mathbf{U_0}=(\mathbf{u_1},\cdots,\mathbf{u_n})$. By (10), we have

$$(z_1,\cdots,z_n) = (f_1(z),\cdots,f_n(z))\cdot\mathbf{U_0}. \tag{11}$$

By the singular value decomposition of symmetric matrices, there exists an unitary $n\times n$ matrix $\mathbf{V}$ such that

$$\mathbf{V}^t\mathbf{U_0}^t\mathbf{U_0}\mathbf{V} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$$

for real numbers λ_i satisfying $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Apply the unitary change of coordinate in $\mathbb{C}^n$ by letting $\widetilde{Z} = (\widetilde{z}_1, \cdots, \widetilde{z}_n) = Z \cdot \mathbf{V}$, where $Z = (z_1, \cdots, z_n)$ or equivalently, $Z = \widetilde{Z} \cdot \mathbf{V}^{-1}$. By (11), we have

$$(\widetilde{z}_1, \cdots, \widetilde{z}_n) = \left(f_1(\widetilde{Z}\mathbf{V}^{-1}), \cdots, f_{n+1}(\widetilde{Z}\mathbf{V}^{-1})\right) \mathbf{\hat{U}_0},$$

where $\mathbf{\hat{U}_0} = \mathbf{U_0}\mathbf{V}$ with $\mathbf{\hat{U}_0^t}\mathbf{\hat{U}_0} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$. Moreover, by (10), we have

$$\left(\widetilde{z}_1, \cdots, \widetilde{z}_n, \frac{1}{2}\sum_{i=1}^{n+1} f_i^2(\widetilde{Z}\mathbf{V}^{-1})\right) = \left(f_1(\widetilde{Z}\mathbf{V}^{-1}), \cdots, f_{n+1}(\widetilde{Z}\mathbf{V}^{-1})\right) (\mathbf{\hat{U}_0}, \mathbf{u_{n+1}}). \tag{12}$$

Let $\widetilde{F}(\widetilde{Z}) = F(\widetilde{Z}\mathbf{V}^{-1})$ and write $\widetilde{F}(\widetilde{Z}) = (\widetilde{f}_1(\widetilde{Z}), \cdots, \widetilde{f}_{n+1}(\widetilde{Z}))$. Then equation (12) can be rewritten as

$$\left(\widetilde{z}_1, \cdots, \widetilde{z}_n, \frac{1}{2}\sum_{i=1}^{n+1} \widetilde{f}_i^2(\widetilde{Z})\right) = \left(\widetilde{f}_1(\widetilde{Z}), \cdots, \widetilde{f}_{n+1}(\widetilde{Z})\right) (\mathbf{\hat{U}_0}, \mathbf{u_{n+1}}). \tag{13}$$

Note that $(\mathbf{\hat{U}_0}, \mathbf{u_{n+1}})$ is still an unitary $(n+1) \times (n+1)$ matrix. It follows from (13) that $\widetilde{F}$ is also a holomorphic isometry from $\mathbb{B}^n$ to D_{n+1}^{IV} and moreover, $\widetilde{F}$ is equivalent to F.

Second normalization: In the following, we write $\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^k x_i y_i$ for two $k-$dimensional vectors $\mathbf{x} = (x_1, ..., x_k), \mathbf{y} = (y_1, ..., y_k)$. We now consider the new map $\widetilde{F}$ in the new holomorphic coordinate $\widetilde{Z}$. But for the simplicity of notations, we still use $F, Z, \mathbf{U_0}$ to denote $\widetilde{F}, \widetilde{Z}, \mathbf{\hat{U}_0}$ respectively. Therefore, we have

$$\left(z_1, \cdots, z_n, \frac{1}{2}\sum_{j=1}^{m} f_j^2(z)\right) = (f_1(z), \cdots, f_{n+1}(z)) \cdot (\mathbf{U_0}, \mathbf{u_{n+1}})$$

with

$$\mathbf{U_0}^t\mathbf{U_0} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}. \tag{14}$$

Write $\mathbf{U_0} = (\mathbf{u_1}, \cdots, \mathbf{u_n})$. It follows from (14) that

$$\begin{aligned} &\mathbf{u_i} \cdot \mathbf{u_i} = \lambda_i \in \mathbb{R} \text{ for } 1 \leq i \leq n, \\ &\mathbf{u_i} \cdot \mathbf{u_j} = 0 \text{ for } 1 \leq i \neq j \leq n. \end{aligned} \tag{15}$$

Note $\overline{\mathbf{U}}_\mathbf{0}^\mathbf{t}\mathbf{U_0} = \mathbf{I_n}$. We have

$$\begin{aligned} &\mathbf{u_i} \cdot \overline{\mathbf{u_i}} = 1 \text{ for } 1 \leq i \leq n, \\ &\mathbf{u_i} \cdot \overline{\mathbf{u_j}} = 0 \text{ for } 1 \leq i \neq j \leq n. \end{aligned} \tag{16}$$

Write $\mathbf{u_i} = \mathbf{a_i} + \sqrt{-1}\mathbf{b_i}$ for $1 \le i \le n$. It follows from (15)-(16) that

$$\begin{aligned} \mathbf{a_i} \cdot \mathbf{b_j} &= 0 \text{ for } 1 \le i, j \le n, \\ \mathbf{a_i} \cdot \mathbf{a_j} &= 0 \text{ for } 1 \le i \ne j \le n, \\ \mathbf{b_i} \cdot \mathbf{b_j} &= 0 \text{ for } 1 \le i \ne j \le n, \\ \mathbf{a_i} \cdot \mathbf{a_i} + \mathbf{b_i} \cdot \mathbf{b_i} &= 1 \text{ for } 1 \le i \le n. \end{aligned} \tag{17}$$

Therefore, these $2n$ vectors $\{\mathbf{a_i}, \mathbf{b_i}\}_{\mathbf{1 \le i \le n}}$ in $\mathbb{R}^{n+1}$ are mutually orthogonal. This implies that at least $n-1$ of them are zero vectors. However, by the last equation in (17), $\mathbf{a_i}$ and $\mathbf{b_i}$ cannot be both zero for each $1 \le i \le n$. Hence, by applying again the unitary change of coordinates in $\mathbb{C}^n$ if necessary, we assume that for each $1 \le j \le n-1$, either $\mathbf{a_j}$ or $\mathbf{b_j}$ is zero. Furthermore, for each fixed $1 \le j \le n-1$, by applying the unitary change of coordinates

$$(z_1, \cdots, z_{j-1}, z_j, z_{j+1} \cdots, z_n) \to (z_1, \cdots, z_{j-1}, \sqrt{-1}z_j, z_{j+1} \cdots, z_n)$$

in $\mathbb{C}^n$ if necessary, we can always assume that $\mathbf{b_j} = 0, \mathbf{a_j} \ne 0$ for $1 \le j \le n-1$. Therefore, we have $\mathbf{u_j} = \mathbf{a_j} \in \mathbb{R}^{\mathbf{n+1}}$ for all $1 \le j \le n-1$, and moreover,

$$(\mathbf{u_1}, \cdots, \mathbf{u_{n-1}})^{\mathbf{t}}(\mathbf{u_1}, \cdots, \mathbf{u_{n-1}}) = \mathbf{I_{n-1}}.$$

Extend $\{\mathbf{u_1}, \cdots, \mathbf{u_{n-1}}\}$ to an orthonormal basis $\{\mathbf{u_1}, \cdots, \mathbf{u_{n-1}}, \mathbf{c_n}, \mathbf{c_{n+1}}\}$ of $\mathbb{R}^{n+1}$ and write the $(n+1) \times (n+1)$ matrix

$$\mathbf{C} = (\mathbf{u_1}, \cdots, \mathbf{u_{n-1}}, \mathbf{c_n}, \mathbf{c_{n+1}}).$$

It follows that $\mathbf{C}, \mathbf{C^t}$ are orthogonal matrices $SO(n+1)$ as $\mathbf{CC^t} = \mathbf{C^tC} = \mathbf{I_{n+1}}$. Define $\widetilde{F} = (\widetilde{f}_1, \cdots, \widetilde{f}_{n+1}) = F(\mathbf{C^t})^{-1}$ or equivalently $F = \widetilde{F}\mathbf{C^t}$. One can easily check that $\widetilde{F}$ is still a holomorphic isometry from $\mathbb{B}^n$ into D^{IV}_{n+1} and $\widetilde{F}$ is equivalent to F. Furthermore, one has

$$\left(z_1, \cdots, z_n, \frac{1}{2}\sum_{j=1}^{n+1} \widetilde{f}_j^2\right) = (\widetilde{f}_1, \cdots, \widetilde{f}_{n+1})\mathbf{C^t}(\mathbf{U_0}, \mathbf{u_{n+1}}).$$

Since $\mathbf{C}$ is an orthogonal matrix, then

$$\mathbf{C^tU_0} = (\mathbf{X}, \widetilde{\mathbf{u}}_\mathbf{n})_{\mathbf{(n+1)\times n}},$$

where $\mathbf{X} = \begin{bmatrix} \mathbf{I_{n-1}} \\ \mathbf{0_{2\times(n-1)}} \end{bmatrix}$ and $\mathbf{0_{2\times(n-1)}}$ is the $2 \times (n-1)$ zero matrix. Note

$$\overline{(\mathbf{C^tU_0})^{\mathbf{t}}}(\mathbf{C^tU_0}) = \overline{\mathbf{U}}_\mathbf{0}^{\mathbf{t}}\mathbf{U_0} = \mathbf{I_n},$$

i.e. the columns of $\mathbf{C^tU_0}$ are orthonormal basis in $\mathbb{C}^{n+1}$. Then we conclude that

$$\widetilde{\mathbf{u}}_\mathbf{n} = \left[0, \cdots, 0, \xi_1, \xi_2\right]^{\mathbf{t}}$$

for $\xi_1, \xi_2 \in \mathbb{C}$ with $|\xi_1|^2 + |\xi_2|^2 = 1$.

Summarize: Note that $\mathbf{C^t}(\mathbf{U_0}, \mathbf{u_{n+1}})$ is an $(n+1)\times(n+1)$ unitary matrix. We will again use $F, \mathbf{U}$ to denote $\widetilde{F}, \mathbf{C^t}(\mathbf{U_0}, \mathbf{u_{n+1}})$ respectively for the simplicity of notations. To summarize, we have normalize the original holomorphic isometry to the map F satisfying

$$\left(z_1, \cdots, z_n, \frac{1}{2}\sum_{j=1}^{n+1} f_j^2(z)\right) = (f_1(z), \cdots, f_{n+1}(z))\,\mathbf{U}, \tag{18}$$

where $\mathbf{U}$ is $(n+1)\times(n+1)$ unitary matrix and

$$\mathbf{U} = (\mathbf{X}, \mathbf{u_n}, \mathbf{u_{n+1}})$$

for $\mathbf{X} = \begin{bmatrix} \mathbf{I_{n-1}} \\ \mathbf{0_{2\times(n-1)}} \end{bmatrix}$, $\mathbf{u_n} = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \xi_1 \\ \xi_2 \end{bmatrix}$ and $\mathbf{u_{n+1}} = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \eta_1 \\ \eta_2 \end{bmatrix}$ with

$$|\xi_1|^2 + |\xi_2|^2 = 1, |\eta_1|^2 + |\eta_2|^2 = 1 \text{ and } \xi_1\bar{\eta}_1 + \xi_2\bar{\eta}_2 = 0.$$

Replacing F by $\widetilde{F} = e^{-\sqrt{-1}\alpha}F$ and writing $\widetilde{F} = (\widetilde{f}_1, \cdots, \widetilde{f}_{n+1})$, we have:

$$\left(z_1, \cdots, z_n, \frac{1}{2}\sum_{j=1}^{n+1} \widetilde{f}_j^2(z)\right) = \left(\widetilde{f}_1(z), \cdots, \widetilde{f}_{n+1}(z)\right) \cdot \begin{bmatrix} e^{\sqrt{-1}\alpha}\mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & e^{\sqrt{-1}\alpha}\xi_1 & e^{-\sqrt{-1}\alpha}\eta_1 \\ \mathbf{0^t_{(n-1)\times 1}} & e^{\sqrt{-1}\alpha}\xi_2 & e^{-\sqrt{-1}\alpha}\eta_2 \end{bmatrix}.$$

Choose a suitable α such that the real and imaginary parts of $(e^{-\sqrt{-1}\alpha}\eta_1, e^{-\sqrt{-1}\alpha}\eta_2)^t$ are orthoganal. Applying the unitary change of coordinates $\hat{Z} = (\hat{z}_1, \cdots, \hat{z}_n) = e^{-\sqrt{-1}\alpha}(z_1, \cdots, z_n)$ and defining $\hat{F}(\hat{Z}) = (\hat{f}_1(\hat{Z}), \cdots, \hat{f}_{n+1}(\hat{Z})) = \widetilde{F}(e^{\sqrt{-1}\alpha}\hat{Z})$, then one can easily check that $\hat{F}$ satisfies:

$$\left(\hat{z}_1, \cdots, \hat{z}_n, \frac{1}{2}\sum_{j=1}^{n+1} \hat{f}_j^2(\hat{Z})\right) = \left(\hat{f}_1(\hat{Z}), \cdots, \hat{f}_{n+1}(\hat{Z})\right) \cdot \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & \xi_1 & e^{-\sqrt{-1}\alpha}\eta_1 \\ \mathbf{0^t_{(n-1)\times 1}} & \xi_2 & e^{-\sqrt{-1}\alpha}\eta_2 \end{bmatrix}. \tag{19}$$

We will still use F, z, η_i to denote $\hat{F}, \hat{Z}, e^{-\sqrt{-1}\alpha}\eta_i$ for $i = 1, 2$ respectively. Then (19) reads

$$\left(z_1, \cdots, z_n, \frac{1}{2}\sum_{j=1}^{n+1} f_j^2(z)\right) = (f_1(z), \cdots, f_{n+1}(z)) \cdot \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & \xi_1 & \eta_1 \\ \mathbf{0^t_{(n-1)\times 1}} & \xi_2 & \eta_2 \end{bmatrix}$$

with

$$|\xi_1|^2+|\xi_2|^2=1, |\eta_1|^2+|\eta_2|^2=1 \text{ and } \xi_1\bar{\eta}_1+\xi_2\bar{\eta}_2=0, \mathrm{Re}(\eta_1,\eta_2)^t \perp \mathrm{Im}(\eta_1,\eta_2)^t.$$

By applying an automorphism

$$\begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 2}} \\ \mathbf{0_{2\times(n-1)}} & \mathbf{V_{2\times 2}} \end{bmatrix}$$

of D_{n+1}^{IV} with a suitable $\mathbf{V}\in O(2)$, we can further make $\eta_1\in\mathbb{R}$ and $\eta_2=\sqrt{-1}\eta$ for some $\eta\in\mathbb{R}$. By further applying an automorphism

$$\begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0_{1\times(n-1)}} & \pm 1 & 0 \\ \mathbf{0_{1\times(n-1)}} & 0 & \pm 1 \end{bmatrix}$$

of D_{n+1}^{IV} if necessary, we can assume that $\eta_1\geq 0$ and $\eta\geq 0$. Since $\eta_1^2+\eta^2=|\eta_1|^2+|\eta_2|^2=1$, write $\eta_1=\cos\theta, \eta=\sin\theta$ for $\theta\in[0,\pi/2]$ and then $\eta_2=\sqrt{-1}\sin\theta$. As $|\xi_1|^2+|\xi_2|^2=1, \xi_1\bar{\eta}_1+\xi_2\bar{\eta}_2=0$, write $\xi_1=\sqrt{-1}\sin\theta e^{\sqrt{-1}\alpha}, \xi_2=\cos\theta e^{\sqrt{-1}\alpha}$ for $\alpha\in[0,2\pi)$. By applying an unitary transform $(\widetilde{z}_1,\cdots,\widetilde{z}_{n-1},\widetilde{z}_n)=(z_1,\cdots,z_{n-1},e^{\sqrt{-1}\alpha}z_n)$ in $\mathbb{C}^n$ if necessary, we may let $\xi_1=\sqrt{-1}\sin\theta, \xi_2=\cos\theta$. Therefore, we normalize the map $F=(f_1,\cdots,f_{n+1})$ satisfying

$$\left(z_1,\cdots,z_n,\frac{1}{2}\sum_{j=1}^{n+1}f_j^2(z)\right)=(f_1(z),\cdots,f_{n+1}(z))\cdot\begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & \sqrt{-1}\sin\theta & \cos\theta \\ \mathbf{0^t_{(n-1)\times 1}} & \cos\theta & \sqrt{-1}\sin\theta \end{bmatrix}$$

for $\theta\in[0,\pi/2]$. Denote the matrix

$$\mathbf{U}=\begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & \sqrt{-1}\sin\theta & \cos\theta \\ \mathbf{0^t_{(n-1)\times 1}} & \cos\theta & \sqrt{-1}\sin\theta \end{bmatrix} \tag{20}$$

We the proceed in two different cases.

Case I: If $\theta=\pi/4$. In this case, we have

$$\left(z_1,\cdots,z_n,\frac{1}{2}\sum_{j=1}^{n+1}f_j^2(z)\right)=(f_1(z),\cdots,f_{n+1}(z))\cdot\mathbf{U},$$

where

$$\mathbf{U}=\begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & \frac{\sqrt{-2}}{2} & \frac{\sqrt{2}}{2} \\ \mathbf{0^t_{(n-1)\times 1}} & \frac{\sqrt{2}}{2} & \frac{\sqrt{-2}}{2} \end{bmatrix}.$$

By replacing F by

$$F\cdot\begin{bmatrix} -\sqrt{-1}\mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0_{1\times(n-1)}} & \sqrt{-1} & 0 \\ \mathbf{0_{1\times(n-1)}} & 0 & -\sqrt{-1} \end{bmatrix},$$

and then apply the unitary transformation in $\mathbb{C}^n : (\widetilde{z_1}, \cdots, \widetilde{z_n}) = (-\sqrt{-1}z_1, \cdots, \sqrt{-1}z_n)$, we are able to make

$$\mathbf{U} = \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & -\frac{\sqrt{-2}}{2} & \frac{\sqrt{-2}}{2} \\ \mathbf{0^t_{(n-1)\times 1}} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

By solving this linear system, one obtains the map R_n^{IV} in (6).

Case II: If $\theta \in [0, \pi/2]$ with $\theta \neq \pi/4$. In the sequel, we will show that we can actually choose $\theta \in [0, \pi/4]$. For any $\theta \in (\pi/4, \pi/2]$, write $\beta = \pi/2 - \theta \in [0, \pi/4)$. Then

$$\mathbf{U} = \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & \sqrt{-1}\cos\beta & \sin\beta \\ \mathbf{0^t_{(n-1)\times 1}} & \sin\beta & \sqrt{-1}\cos\beta \end{bmatrix}.$$

By applying the automorphism

$$(\widetilde{w}_1, \cdots, \widetilde{w}_{n+1}) = (w_1, \cdots, w_{n+1}) \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & 0 & 1 \\ \mathbf{0^t_{(n-1)\times 1}} & 1 & 0 \end{bmatrix}$$

of D_{n+1}^{IV}, we may let

$$\mathbf{U} = \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & \sin\beta & \sqrt{-1}\cos\beta \\ \mathbf{0^t_{(n-1)\times 1}} & \sqrt{-1}\cos\beta & \sin\beta \end{bmatrix}.$$

Applying the automorphism $(\widetilde{w}_1, \cdots, \widetilde{w}_{n+1}) = -\sqrt{-1}(w_1, \cdots, w_{n+1})$ of D_{n+1}^{IV} and then applying the unitary transform $(\widetilde{z}_1, \cdots, \widetilde{z}_{n-1}, \widetilde{z}_n) = (\sqrt{-1}z_1, \cdots, \sqrt{-1}z_{n-1}, -z_n)$ of $\mathbb{C}^n$, we may let

$$\mathbf{U} = \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & \sqrt{-1}\sin\beta & \cos\beta \\ \mathbf{0^t_{(n-1)\times 1}} & -\cos\beta & -\sqrt{-1}\sin\beta \end{bmatrix}.$$

Finally applying the automorphism $(\widetilde{w}_1, \cdots, \widetilde{w}_{n-1}, \widetilde{w}_{n+1}) = (w_1, \cdots, w_{n-1}, -w_{n+1})$ of D_{n+1}^{IV},

$$\mathbf{U} = \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 1}} & \mathbf{0_{(n-1)\times 1}} \\ \mathbf{0^t_{(n-1)\times 1}} & \sqrt{-1}\sin\beta & \cos\beta \\ \mathbf{0^t_{(n-1)\times 1}} & \cos\beta & \sqrt{-1}\sin\beta \end{bmatrix}.$$

This is the matrix in (20). By solving the system

$$\left(z_1, \cdots, z_n, \frac{1}{2}\sum_{j=1}^{n+1} f_j^2(z)\right) = (f_1(z), \cdots, f_{n+1}(z)) \cdot \mathbf{U},$$

we obtain that F is equivalent to $I_{n,\beta}$ in (7) for some $\beta \in [0, \pi/4)$. This establishes Theorem 3.4. $\square$

Theorem 3.5. *For any $\theta \in [0, \pi/4)$, $I_{n,\theta} : \mathbb{B}^n \to D^{IV}_{n+1}$ given in (7) is equivalent to $I_{n,0}$.*

Proof. We first apply the Borel embedding to embed $\mathbb{B}^n$ as an open subset of $\mathbb{P}^n$ and D^{IV}_{n+1} as an open subset of $\mathbb{Q}^{n+1} \subset \mathbb{P}^{n+2}$, where the Borel embedding is given by

$$z = (z_1, \cdots, z_{n+1}) \in D^{IV}_{n+1} \to \left[z_1, \cdots, z_{n+1}, \frac{1 + \frac{1}{2} z z^t}{\sqrt{2}}, \frac{(1 - \frac{1}{2} z z^t)}{\sqrt{-2}} \right] \in \mathbb{Q}^{n+1} \subset \mathbb{P}^{n+2}.$$

We write $[z, s] = [z_1, \cdots, z_n, s]$ to denote the homogeneous coordinates in $\mathbb{P}^n$. Then under homogeneous coordinates, $I_{n,\theta}$ is identified with

$$\mathcal{I}_{n,\theta}(z, s) = [z_1, \cdots, z_{n-1}, \phi_{n,\theta}(z, s), \phi_{n+1,\theta}(z, s), \phi_{n+2,\theta}(z, s), \phi_{n+3,\theta}(z, s)] \tag{21}$$

from $\mathbb{P}^n$ to $\mathbb{P}^{n+2}$ where

$$\begin{aligned}
\phi_{n,\theta}(z, s) &= \frac{(\cos\theta s + \sqrt{-1} \sin\theta z_n) - \cos\theta \sqrt{H_\theta(z, s)}}{\cos(2\theta)}; \\
\phi_{n+1,\theta}(z, s) &= \frac{(-\sqrt{-1} \sin\theta s + \cos\theta z_n) + \sqrt{-1} \sin\theta \sqrt{H_\theta(z, s)}}{\cos(2\theta)}; \\
\phi_{n+2,\theta}(z, s) &= \frac{1 + \cos(2\theta)}{\sqrt{2}\cos(2\theta)} s + \frac{\sqrt{-1}\tan(2\theta) z_n}{\sqrt{2}} - \frac{1}{\sqrt{2}\cos(2\theta)} \sqrt{H_\theta(z, s)}; \\
\phi_{n+3,\theta}(z, s) &= \frac{\cos(2\theta) - 1}{\sqrt{-2}\cos(2\theta)} s - \frac{\sqrt{-1}\tan(2\theta) z_n}{\sqrt{2}} + \frac{1}{\sqrt{2}\cos(2\theta)} \sqrt{H_\theta(z, s)}; \\
H_\theta(z, s) &= s^2 + 2\sqrt{-1} \sin(2\theta) z_n s - z_n^2 - \cos(2\theta) \sum_{j=1}^{n-1} z_j^2.
\end{aligned} \tag{22}$$

In particular,

$$\mathcal{I}_{n,0}(z, s) = \left[z_1, \cdots, z_{n-1}, s - \sqrt{H_0}, z_n, \frac{2s - \sqrt{H_0}}{\sqrt{2}}, \frac{\sqrt{H_0}}{\sqrt{-2}} \right].$$

Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\cos\theta}{\sqrt{\cos(2\theta)}} & \frac{-\sin\theta}{\sqrt{\cos(2\theta)}}\sqrt{-1} \\ \mathbf{0} & \frac{\sin\theta}{\sqrt{\cos(2\theta)}}\sqrt{-1} & \frac{\cos\theta}{\sqrt{\cos(2\theta)}} \end{bmatrix} \in U(n, 1) = \mathrm{Aut}(\mathbb{B}^{\mathrm{n}}),$$

and define $\hat{\mathcal{I}}_{n,\theta}(z, s) = \mathcal{I}_{n,\theta}\left((z, s) \cdot \mathbf{B}\right)$. Then it follows from the straightforward calculation that

$$\hat{\mathcal{I}}_{n,\theta} = \left[z_1, \cdots, z_{n-1}, \frac{s - \cos\theta\sqrt{H_0}}{\sqrt{\cos(2\theta)}}, \frac{z_n + \sin\theta\sqrt{-H_0}}{\sqrt{\cos(2\theta)}}, \frac{2\cos\theta s - \sqrt{H_0}}{\sqrt{2\cos(2\theta)}}, \frac{-2\sin\theta z_n - \sqrt{-H_0}}{\sqrt{2\cos(2\theta)}} \right].$$

Let $T = \begin{bmatrix} \mathbf{I_{n-1}} & \mathbf{0_{(n-1)\times 4}} \\ \mathbf{0_{4\times(n-1)}} & \mathbf{V} \end{bmatrix}$ with

$$\mathbf{V} = \frac{1}{\sqrt{\cos(2\theta)}} \begin{bmatrix} 1-4\sin^2(\theta/2) & 0 & 2\sqrt{2}\sin^2(\theta/2) & 0 \\ 0 & 1 & 0 & -\sqrt{2}\sin\theta \\ 2\sqrt{2}\sin^2(\theta/2) & 0 & 1-4\sin^2(\theta/2) & 0 \\ 0 & -\sqrt{2}\sin\theta & 0 & 1 \end{bmatrix}.$$

Then one can verify that $T \in \mathrm{Aut}(D_{n+1}^{IV})$ and

$$\begin{aligned} &\mathcal{I}_{n,0} \cdot T \\ &= \left(z_1, \cdots, z_{n-1}, s - \sqrt{H_0}, z_n, \frac{2s - \sqrt{H_0}}{\sqrt{2}}, \frac{\sqrt{H_0}}{\sqrt{-2}} \right) \cdot T \\ &= \left(z_1, \cdots, z_{n-1}, \frac{s - \cos\theta\sqrt{H_0}}{\sqrt{\cos(2\theta)}}, \frac{z_n + \sin\theta\sqrt{-H_0}}{\sqrt{\cos(2\theta)}}, \frac{2\cos\theta s - \sqrt{H_0}}{\sqrt{2\cos(2\theta)}}, \frac{-2\sin\theta z_n - \sqrt{-H_0}}{\sqrt{2\cos(2\theta)}} \right) \\ &= \hat{\mathcal{I}}_{n,\theta} = \mathcal{I}_{n,\theta}\left((z,s)\cdot \mathbf{B}\right). \end{aligned}$$

This implies that $I_{n,\theta}$ is equivalent to $I_{n,0}$ for any $\theta \in [0, \pi/4)$ by the discussion in section 4.1. □

Note that a rational map from $\mathbb{B}^n \to D_{n+1}^{IV}$ cannot be equivalent to an irrational map. Thus combining Theorem 3.4, 3.5 we obtain the classification result Theorem 1.1 for holomorphic isometries from $\mathbb{B}^n$ into D_{n+1}^{IV}.

3.3 Minimal holomorphic isometries into Type IV domains

Let F be a holomorphic isometry from $\mathbb{B}^n$ to $D_{m'}^{IV}$ for $m' \geq n+1 \geq 2$. Let $m > m'$. Then $(F, \mathbf{0})$, where $\mathbf{0}$ is a $(m-m')$-dimensional zero row vector, is a holomorphic isometry from $\mathbb{B}^n$ to D_m^{IV}. We introduce the following definition.

Definition 3.6. *A holomorphic isometry $F : \mathbb{B}^n \to D_m^{IV}$ is minimal if there is no holomorphic isometry $G : \mathbb{B}^n \to D_{m-1}^{IV}$ such that F is equivalent to $(G, 0)$.*

Remark 3.7. *It follows from Mok's theorem (Theorem 2.1(i)) that any holomorphic isometry $F : \mathbb{B}^n \to D_{n+1}^{IV}$ is minimal.*

We prove the following result.

Theorem 3.8. *Let $m > 2n+2$. Let $F : \mathbb{B}^n \to D_m^{IV}$ be a holomorphic isometry. Then there exists a holomorphic isometry $G : \mathbb{B}^n \to D_{2n+2}^{IV}$ such that F is equivalent to $(G, \mathbf{0})$, where $\mathbf{0}$ is a $(m-2n-2)$-dimensional zero row vector.*

Proof. It suffices to show that for any such F and m, there exists a holomorphic isometry $\hat{F} : \mathbb{B}^n \to D^{IV}_{m-1}$ such that F is equivalent to $(\hat{F}, \mathbf{0})$. Write $F = (F_1, \cdots, F_m)$. By the isometry assumption, we have

$$(F_1, \cdots, F_m) = (z_1, \cdots, z_n, \frac{1}{2}\sum_{i=1}^m F_i^2(z), 0, \cdots, 0) \cdot \mathbf{V},$$

where $\mathbf{V}$ is an $m \times m$ unitary matrix. Write

$$V = \begin{bmatrix} \mathbf{v_1} \\ \cdots \\ \mathbf{v_m} \end{bmatrix},$$

where $\mathbf{v_i}$ is a m-dimensional row vector for $1 \leq i \leq m$. Then we have

$$\left(z_1, \cdots, z_n, \frac{1}{2}\sum_{i=1}^m F_i^2(z) \right) \cdot \begin{bmatrix} \mathbf{v_1} \\ \cdots \\ \mathbf{v_{n+1}} \end{bmatrix} = (F_1, \cdots, F_m).$$

Claim: $\{F_1, \cdots, F_m\}$ is a linearly dependent set over real number field. In other words, there exist $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$ not mutually zero, such that $\sum_{i=1}^m \lambda_i F_i \equiv 0$.

Proof of Claim: Write $\mathbf{v_i} = \mathbf{a_i} + \sqrt{-\mathbf{1}}\mathbf{b_i}$ for $1 \leq i \leq n+1$. It is easy to see that there exists $\lambda = (\lambda_1, \cdots, \lambda_m)^t \in \mathbb{R}^m$ with $\lambda \neq \mathbf{0}$ such that

$$\begin{bmatrix} \mathbf{a_1} \\ \mathbf{b_1} \\ \cdots \\ \mathbf{a_{n+1}} \\ \mathbf{b_{n+1}} \end{bmatrix} \lambda = \mathbf{0}. \tag{23}$$

This is because

$$\text{rank} \begin{bmatrix} \mathbf{a_1} \\ \mathbf{b_1} \\ \cdots \\ \mathbf{a_{n+1}} \\ \mathbf{b_{n+1}} \end{bmatrix} \leq 2(n+1) < m.$$

Then (23) implies

$$\left(z_1, \cdots, z_n, \frac{1}{2}\sum_{i=1}^m F_i^2(z) \right) \cdot \begin{bmatrix} \mathbf{v_1} \\ \cdots \\ \mathbf{v_{n+1}} \end{bmatrix} \cdot \lambda = \mathbf{0}.$$

This proves the claim by showing $\sum_{i=1}^m \lambda_i F_i \equiv 0$.

By rescaling λ if necessary, we assume $|\lambda| = \mathbf{1}$. Extend λ to an orthonormal basis $\{\mathbf{u_1}, \cdots, \mathbf{u_{m-1}}, \lambda\}$ of $\mathbb{R}^m$ and write $m \times m$ matrix $\mathbf{C} = (\mathbf{u_1}, \cdots, \mathbf{u_{m-1}}, \lambda)$. Define $\hat{F} = (\hat{F}_1, \cdots, \hat{F}_m) = F \cdot \mathbf{C}$ and then $\hat{F}$ is equivalent to F. This completes the proof of the theorem because $\hat{F}_m = F \cdot \lambda = \mathbf{0}$. $\square$

We then study holomorphic isometries from $\mathbb{B}^n$ to D_m^{IV} with $n+2 \le m \le 2n+2$. Define $G_{n+2}, H_{n+2} : \mathbb{B}^n \to D_{n+2}^{IV}$ to be

$$G_{n+2}(z) = \Bigg(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, z_2, \cdots, z_{n-1},$$

$$\frac{\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n-1} z_j^2 - 2z_n^2 + 2z_n}{2\sqrt{2}(1-z_n)}, \frac{\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n-1} z_j^2 + 2z_n^2 - 2z_n}{2\sqrt{-2}(1-z_n)} \Bigg),$$

$$H_{n+2}(z) = \left(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, z_2, \cdots, z_n, 1 - \sqrt{1 - \cos(2\theta_1)z_1^2 - \sum_{j=2}^{n} z_j^2} \right)$$

with $\theta \in (0, \pi/4]$; and one can similarly G_{n+k}, H_{n+k} for $2 \le k \le n$ up to $G_{2n}, H_{2n} : \mathbb{B}^n \to D_{2n}^{IV}$ given by

$$G_{2n}(z) = \Bigg(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_{n-1} z_{n-1}, \sqrt{-1}\sin\theta_{n-1} z_{n-1},$$

$$\frac{\sum_{j=1}^{n-1} \cos(2\theta_j)z_j^2 - 2z_n^2 + 2z_n}{2\sqrt{2}(1-z_n)}, \frac{\sum_{j=1}^{n-1} \cos(2\theta_j)z_j^2 + 2z_n^2 - 2z_n}{2\sqrt{-2}(1-z_n)} \Bigg),$$

$$H_{2n}(z) = \Bigg(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_{n-1} z_{n-1}, \sqrt{-1}\sin\theta_{n-1} z_{n-1},$$

$$z_n, 1 - \sqrt{1 - \sum_{j=1}^{n-1} \cos(2\theta_j)z_j^2 - z_n^2} \Bigg)$$

with $\theta_j \in (0, \pi/4]$ for $1 \le j \le n-1$. Furthermore, $G_{2n+1}, H_{2n+1} : \mathbb{B}^n \to D_{2n+1}^{IV}$ are given by

$$G_{2n+1}(z) = \Bigg(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_{n-1} z_{n-1}, \sqrt{-1}\sin\theta_{n-1} z_{n-1}, z_n,$$

$$\frac{1}{2\sqrt{2}} \left(\sum_{j=1}^{n-1} \cos(2\theta_j)z_j^2 + z_n^2 \right), \frac{-\sqrt{-1}}{2\sqrt{2}} \left(\sum_{j=1}^{n-1} \cos(2\theta_j)z_j^2 + z_n^2 \right) \Bigg)$$

for $\theta_1, \cdots, \theta_{n-1} \in (0, \pi/4]$,

$$H_{2n+1}(z) = \left(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_n z_n, \sqrt{-1}\sin\theta_n z_n, 1 - \sqrt{1 - \sum_{j=1}^{n} \cos(2\theta_j)z_j^2} \right)$$

for $\theta_1, \cdots, \theta_n \in (0, \pi/4]$ but not all $\theta_j = \pi/4$; and $G_{2n+2}, H_{2n+2} : \mathbb{B}^n \to D_{2n+2}^{IV}$ are given by

$$G_{2n+2}(z) = \Bigg(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_n z_n, \sqrt{-1}\sin\theta_n z_n,$$

$$\frac{1}{2\sqrt{2}}\sum_{j=1}^{n}\cos(2\theta_j)z_j^2, \frac{-\sqrt{-1}}{2\sqrt{2}}\sum_{j=1}^{n}\cos(2\theta_j)z_j^2\Bigg),$$

$$H_{2n+2}(z) = \Bigg(\cos\theta_1 z_1, \sqrt{-1}\sin\theta_1 z_1, \cdots, \cos\theta_n z_n, \sqrt{-1}\sin\theta_n z_n,$$

$$\frac{\cos\theta}{\cos(2\theta)}\left(1-\sqrt{1-\cos(2\theta)\left(\sum_{j=1}^{n}\cos(2\theta_j)z_j^2\right)}\right), \frac{\sqrt{-1}\sin\theta}{\cos(2\theta)}\left(1-\sqrt{1-\cos(2\theta)\left(\sum_{j=1}^{n}\cos(2\theta_j)z_j^2\right)}\right)\Bigg)$$

for $\theta_j \in (0, \pi/4]$ but not all $\theta_j = \pi/4$ for $1 \leq j \leq n$ and $\theta \in (0, \pi/4)$.

Theorem 3.9. *For each $2 \leq k \leq n+2$, $G_{n+k}, H_{n+k} : \mathbb{B}^n \to D^{IV}_{n+k}$ are minimal.*

Proof. We will only prove the case $k = 2$ and other cases follow by similar argument. Apply the Borel embedding to embed $\mathbb{B}^n$ as an open subset of $\mathbb{P}^n$ and D^{IV}_m as an open subset of $\mathbb{Q}^m \subset \mathbb{P}^{m+1}$ as before and write $[z, s] = [z_1, \cdots, z_n, s]$ to denote the homogeneous coordinates in $\mathbb{P}^n$.

We first prove the theorem for G_{n+2}. Under the homogeneous coordinates, G_{n+2} can be identified with

$$\mathcal{G}_{n+2}(z, s) = [g_1(z, s), \cdots, g_{n+3}(z, s)]$$

from $\mathbb{P}^n$ to $\mathbb{P}^{n+3}$, where

$$\begin{aligned}
g_1(z, s) &= \cos\theta_1(s - z_n)z_1;\\
g_2(z, s) &= \sqrt{-1}\sin\theta_1(s - z_n)z_1;\\
g_j(z, s) &= (s - z_n)z_{j-1} \text{ for } 3 \leq j \leq n;\\
g_{n+1}(z, s) &= \frac{\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n-1} z_j^2 - 2z_n^2 + 2z_n s}{2\sqrt{2}};\\
g_{n+2}(z, s) &= \frac{\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n-1} z_j^2 + 2z_n^2 - 2z_n s}{2\sqrt{-2}};\\
g_{n+3}(z, s) &= \frac{1}{2\sqrt{2}}\left(\cos(2\theta_1)z_1^2 + \sum_{j=2}^{n} z_j^2 + 2s^2 - 2z_n s\right);\\
g_{n+4}(z, s) &= \frac{1}{2\sqrt{-2}}\left(-\cos(2\theta_1)z_1^2 - \sum_{j=2}^{n} z_j^2 + 2s^2 - 2z_n s\right).
\end{aligned}$$

Claim: The set $\{g_1, \cdots, g_{n+4}\}$ is linearly independent over $\mathbb{R}$ on any open subset of $\mathbb{C}^{n+1}$. Consequently, for any $\mathbf{B} \in U(n, 1)$, the set $\{\hat{g}_1, \cdots, \hat{g}_{n+4}\}$ with $\hat{g}_j = g_j((z, s) \cdot \mathbf{B})$ is linearly independent over $\mathbb{R}$.

Proof of Claim: We will just prove the first part of the claim and the second part follows easily. Let $\{a_1, \cdots, a_{n+4}\}$ be a set of real numbers such that $\sum_{j=1}^{n+4} a_j\hat{g}_j \equiv 0$. By comparing coefficients of $z_n z_j$ for $1 \le j \le n-1$, we know $a_j = 0$ for $1 \le j \le n$. By comparing coefficients of z_n^2, we know $a_{n+1} = a_{n+2} = 0$. By comparing coefficients of s^2, we know $a_{n+3} = a_{n+4} = 0$. This proves the claim.

Now suppose that G_{n+2} is not minimal. Namely, there exists $F : \mathbb{B}^n \to D_m^{IV}$ with $m < n+2$ such that G_{n+2} is equivalent to $(F, 0)$. More precisely, under homogeneous coordinates, there exist $\mathbf{B} \in U(n,1)$ and $T \in \mathrm{Aut}(D_{n+2}^{IV})$ such that

$$\mathcal{G}_{n+2}((z,s) \cdot \mathbf{B}) \cdot T = [\tilde{F}(z,s), 0, \cdots], \tag{24}$$

where $\tilde{F}$ is the map obtained from F under homogeneous coordinates. By comparing the $(n+2)$-th element in (24), we deduce a contradiction to the claim. This shows that G_{n+2} must be minimal.

The conclusion for H_{n+2} in the theorem follows from the similar argument. Under the homogeneous coordinates, H_{n+2} can be identified with

$$\mathcal{H}_{n+2}(z,s) = [h_1(z,s), \cdots, h_{n+4}(z,s)]$$

from $\mathbb{P}^n$ to $\mathbb{P}^{n+3}$, where

$$\begin{aligned}
h_1(z,s) &= \cos\theta_1 z_1; \\
h_2(z,s) &= \sqrt{-1}\sin\theta_1 z_1; \\
h_j(z,s) &= z_{j-1} \text{ for } 3 \le j \le n+1; \\
h_{n+2}(z,s) &= s - \sqrt{s^2 - \cos(2\theta_1) z_1^2 - \sum_{j=2}^{n} z_j^2}; \\
h_{n+3}(z,s) &= \frac{1}{\sqrt{2}}\left(2s - \sqrt{s^2 - \cos(2\theta_1) z_1^2 - \sum_{j=2}^{n} z_j^2}\right); \\
h_{n+4}(z,s) &= \frac{1}{\sqrt{-2}}\sqrt{s^2 - \cos(2\theta_1) z_1^2 - \sum_{j=2}^{n} z_j^2}.
\end{aligned}$$

Claim: The set $\{h_1, \cdots, h_{n+4}\}$ is linearly independent over $\mathbb{R}$ on any open subset of $\mathbb{C}^{n+1}$. Consequently, for any $\mathbf{B} \in U(n,1)$, the set $\{\hat{h}_1, \cdots, \hat{h}_{n+4}\}$ with $\hat{h}_j = h_j((z,s)\cdot\mathbf{B})$ is linearly independent over $\mathbb{R}$.

The proof the claim is very similar to the previous one. Let $\{a_1, \cdots, a_{n+4}\}$ be a set of real numbers such that $\sum_{j=1}^{n+4} a_j \hat{h}_j \equiv 0$. Then one can show $a_j = 0$ for all j by comparing coefficients.

The rest proof of the theorem is also similar. Suppose that H_{n+2} is not minimal. Namely, there exists $F : \mathbb{B}^n \to D_m^{IV}$ with $m < n+2$ such that H_{n+2} is equivalent to $(F, 0)$. More precisely, under homogeneous coordinates, there exist $\mathbf{B} \in U(n,1)$ and $T \in \mathrm{Aut}(D^{IV}_{n+2})$ such that

$$\mathcal{H}_{n+2}((z,s)\cdot \mathbf{B}) \cdot T = (\tilde{F}(z,s), \cdots, 0, \cdots), \tag{25}$$

where $\tilde{F}$ is the map obtained from F under homogeneous coordinates. By comparing the $(n+2)$-th element in (25), we deduce a contradiction. This shows that H_{n+2} must be minimal. □

3.4 Nonequivalent families of holomorphic isometries

Theorem 1.1 states that there are only two equivalent classes of holomorphic isometries in the maximal dimensional case $\mathbb{B}^n$ to D^{IV}_{m+1}. A natural question is then raised to understand the complexity of holomorphic isometries into higher dimensional target space. We will prove that there are infinitely many equivalent classes in any higher dimension.

For $\theta \in (0, \pi/4]$, define

$$I_{n+2,\theta}(z) = \left(\cos\theta z_1, \sqrt{-1}\sin\theta z_1, z_2, \cdots, z_n, 1 - \sqrt{1 - \cos(2\theta) z_1^2 - \sum_{j=2}^n z_j^2} \right).$$

Fixing $\beta \in (0, \pi/4)$, for $\theta \in [\beta, \pi/4]$ define

$$I_{n+3,\theta}(z) = \left(\cos\theta z_1, \sqrt{-1}\sin\theta z_1, \cos\beta z_2, \sqrt{-1}\sin\beta z_2, z_3, \cdots, z_n, 1 - \sqrt{1 - \cos(2\theta) z_1^2 - \sum_{j=2}^n z_j^2} \right).$$

Similarly, we define $I_{n+k,\theta}$ for $2 \le k \le n+1$ up to

$$I_{2n+1,\theta} = \Bigg(\cos\theta z_1, \sqrt{-1}\sin\theta z_1, \cos\beta z_2, \sqrt{-1}\sin\beta z_2, \cdots, \cos\beta z_n, \sqrt{-1}\sin\beta z_n,$$
$$1 - \sqrt{1 - \cos(2\theta) z_1^2 - \cos(2\beta)\sum_{j=2}^n z_j^2} \Bigg)$$

for $0 < \beta \le \theta \le \pi/4$. Fixing $\alpha \in (0, \pi/4)$ and $\beta \in (0, \pi/4)$, we define

$$I_{2n+2,\theta} = \Bigg(\cos\theta z_1, \sqrt{-1}\sin\theta z_1, \cos\beta z_2, \sqrt{-1}\sin\beta z_2, \cdots, \cos\beta z_n, \sqrt{-1}\sin\beta z_n,$$
$$\frac{\cos\alpha}{\cos(2\alpha)} \left(1 - \sqrt{1 - \cos(2\alpha)\left(\cos(2\theta) z_1^2 + \cos(2\beta)\sum_{j=2}^n z_j^2 \right)} \right),$$

$$\frac{\sqrt{-1}\sin\alpha}{\cos(2\alpha)}\left(\sqrt{1-\cos(2\alpha)\left(\cos(2\theta)z_1^2+\cos(2\beta)\sum_{j=2}^n z_j^2\right)}-1\right)\Bigg)$$

for $\beta \le \theta \le \pi/4$. For $n \ge 2, 2 \le k \le n+2$, we will show that there exists a real parameter family of nonequivalent minimal holomorphic isometries from $\mathbb{B}^n$ to D^{IV}_{n+k}.

Theorem 3.10. *Letting $n \ge 2$, then we have*

- *$\{I_{n+2,\theta}\}_{0<\theta\le\pi/4}$ is a family of nonequivalent holomorphic isometries.*
- *For each $2 < k < n+2$ and fixed $\beta \in (0, \pi/4)$, $\{I_{n+k,\theta}\}_{\beta\le\theta\le\pi/4}$ is a family of nonequivalent holomorphic isometries.*
- *For fixed $\beta \in (0, \pi/4)$, $\{I_{2n+2,\theta}\}_{\beta\le\theta\le\pi/4}$ is a family of nonequivalent holomorphic isometries.*

More precisely, for each $2 \le k \le n+2$, I_{n+k,θ_1} is equivalent to I_{n+k,θ_2} if and only if $\theta_1 = \theta_2$.

Proof. We will merely prove the case $k = 2$ and the remaining cases follow by similar argument. Let $0 < \theta_2 < \theta_1 \le \pi/4$. Then we show that I_{k+2,θ_1} and I_{n+2,θ_2} are not equivalent.

Apply the Borel embedding to embed $\mathbb{B}^n$ as an open subset of $\mathbb{P}^n$ and D^{IV}_{n+2} as an open subset of $\mathbb{Q}^{n+2} \subset \mathbb{P}^{n+3}$ as before and write $[z, s] = [z_1, \cdots, z_n, s]$ to denote the homogeneous coordinates in $\mathbb{P}^n$. Under the homogeneous coordinates, $I_{n+2,\theta}$ can be identified with

$$\mathcal{I}_{n+2,\theta}(z, s) = [\phi_{1,\theta}(z, s), \cdots, \phi_{n+4,\theta}(z, s)]$$

from $\mathbb{P}^n$ to $\mathbb{P}^{n+3}$, where

$$\begin{aligned}
\phi_{1,\theta}(z, s) &= \cos\theta z_1;\\
\phi_{2,\theta}(z, s) &= \sqrt{-1}\sin\theta z_1;\\
\phi_{j,\theta}(z, s) &= z_{j-1} \text{ for } 3 \le j \le n+1;\\
\phi_{n+2,\theta}(z, s) &= s - \sqrt{H_\theta(z, s)};\\
\phi_{n+3,\theta}(z, s) &= \frac{1}{\sqrt{2}}\left(2s - \sqrt{H_\theta(z, s)}\right);\\
\phi_{n+4,\theta}(z, s) &= \frac{1}{\sqrt{-2}}\sqrt{H_\theta(z, s)},
\end{aligned}$$

where $H_\theta(z, s) = s^2 - \cos(2\theta)z_1^2 - \sum_{j=2}^n z_j^2$. Note that for any $\theta \in [0, \pi/4)$ and $n \ge 2$, H_θ is irreducible and in particular, H_θ is not a perfect square.

By the previous argument, I_{n+2,θ_1} is equivalent to I_{n+2,θ_2} if and only if there exist $\mathbf{U} \in U(n, 1)$ and $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Aut}(D^{IV}_{n+2})$ such that

$$(\phi_{1,\theta_1}, \cdots, \phi_{n+2,\theta_1})((z, s)\mathbf{U}) \equiv \frac{(\phi_{1,\theta_2}, \cdots, \phi_{n+2,\theta_2})A + (\phi_{n+3,\theta_2}, \phi_{n+4,\theta_2})C}{((\phi_{1,\theta_2}, \cdots, \phi_{n+2,\theta_2})B + (\phi_{n+3,\theta_2}, \phi_{n+4,\theta_2})D)(1/\sqrt{2}, \sqrt{-1/2})^t}(z, s). \tag{26}$$

Consequently, one has

$$\sqrt{H_{\theta_1}((z,s)\mathbf{U})} = R_1(z,s)\sqrt{H_{\theta_2}(z,s)} + R_2(z,s)$$

for rational functions $R_1(z,s), R_2(z,s)$. This is impossible by algebra if the following claim is true.

Claim: For any $\mathbf{U} \in U(n,1)$, $H_{\theta_1}((z,s)\mathbf{U})$ and $H_{\theta_2}(z,s)$ are coprime.

Proof of Claim: Suppose not. Since they are both irreducible, then there exists $\mathbf{U} \in U(n,1)$ such that

$$H_{\theta_1}((z,s)\mathbf{U}) = cH_{\theta_2}(z,s) \tag{27}$$

for some nonzero complex number c. Write $H_\theta = -(z,s)A_\theta(z,s)^t$ with $A_\theta = \mathrm{diag}(\cos(2\theta), 1, \cdots, 1, -1)$. Then (27) yields that

$$\mathbf{U} \cdot A_{\theta_1} \cdot \mathbf{U}^t = c \cdot A_{\theta_2}.$$

This is impossible by Proposition 3.11. This finishes the proof of the claim. □

Proposition 3.11. *Let $\lambda, \lambda_1, \cdots, \lambda_{n+1}$ be real numbers such that $|\lambda| < |\lambda_1| \leq \cdots \leq |\lambda_{n+1}|$ for $n \geq 1$. Then there does not exist an $(n+1) \times (n+1)$ matrix $\mathbf{U} \in U(n,1)$, such that*

$$\mathbf{U} \cdot \mathrm{diag}(\lambda, \lambda_2, \cdots, \lambda_{n+1}) \cdot \mathbf{U}^t = c \cdot \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{n+1}) \tag{28}$$

for some complex number c.

Proof. Write

$$\mathbf{U} = \begin{bmatrix} a_1 & b_1 & c_1 & \cdots \\ a_2 & b_2 & c_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+1} & b_{n+1} & c_{n+1} & \cdots \end{bmatrix}$$

and note that $\{a_1, a_2, \cdots, a_{n+1}\}$ cannot be all zero.

Claim: Only one element in $\{a_1, \cdots, a_{n+1}\}$ is not zero.

Proof of Claim: We will merely present the proof for $n = 3$ and the general case is similar. Suppose that the claim is not true. Then any vector (a_i, a_j, a_k) for $1 \leq i < j < k \leq 4$ is a nonzero vector. We now claim

$$a_4 \cdot \overline{\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}} = -b_4 \cdot \overline{\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}}, \tag{29}$$

$$a_4 \cdot \overline{\det \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}} = c_4 \cdot \overline{\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}} \tag{30}$$

$$a_4 \cdot \det \overline{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}} = d_4 \cdot \det \overline{\begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}} \tag{31}$$

We only prove (29) and two others are similar. Note if both $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}$ and $\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}$ are zero, then (29) holds trivially. Without loss of generality, assume $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix} \neq 0$. The case $\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} \neq 0$ can be proved similarly. It follows from $\mathbf{U} \in U(n,1)$ that

$$(a_4, b_4, c_4, -d_4) \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \\ \bar{d}_1 & \bar{d}_2 & \bar{d}_3 \end{bmatrix} = (0,0,0).$$

This implies that

$$\begin{bmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 & \bar{d}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 & \bar{d}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 & \bar{d}_3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_4 \\ b_4 \\ c_4 \\ -d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_4 \end{bmatrix}.$$

Namely, $(a_4, b_4, c_4, -d_4)^t$ is the solution of the linear system:

$$\begin{bmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 & \bar{d}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 & \bar{d}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 & \bar{d}_3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_4 \end{bmatrix}.$$

By the Cramer's rule, we know

$$a_4 = -b_4 \cdot \det \begin{bmatrix} \bar{b}_1 & \bar{c}_1 & \bar{d}_1 \\ \bar{b}_2 & \bar{c}_2 & \bar{c}_2 \\ \bar{b}_3 & \bar{c}_3 & \bar{d}_3 \end{bmatrix} / \det \begin{bmatrix} \bar{a}_1 & \bar{c}_1 & \bar{d}_1 \\ \bar{a}_2 & \bar{c}_2 & \bar{c}_2 \\ \bar{a}_3 & \bar{c}_3 & \bar{d}_3 \end{bmatrix}.$$

This implies (29).

We further claim:

$$\frac{\lambda}{\lambda_2} a_4 \cdot \det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix} = -b_4 \cdot \det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}, \tag{32}$$

$$\frac{\lambda}{\lambda_3} a_4 \cdot \det \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix} = c_4 \cdot \det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} \tag{33}$$

$$\frac{\lambda}{\lambda_4} a_4 \cdot \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = -d_4 \cdot \det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} \tag{34}$$

We only prove (32) and two others are similar. Note again if both $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}$ and $\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix}$ are zero, then (30) holds trivially. Without loss of generality, assume $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix} \neq 0$. The case $\det \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} \neq 0$ can be proved similarly. It follows from (28) that

$$(\lambda a_4, \lambda_2 b_4, \lambda_3 c_4, \lambda_4 d_4) \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix} = (0, 0, 0).$$

This implies:

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda a_4 \\ \lambda_2 b_4 \\ \lambda_3 c_4 \\ \lambda_4 d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \lambda_2 b_4 \end{bmatrix}.$$

Hence, (32) follows from the Cramer's rule.

Equations (29) and (32) imply that $a_4 \cdot \det \overline{\begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}}$ and $\frac{\lambda}{\lambda_2} a_4 \cdot \det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}$ have the same norm. However, $|\lambda/\lambda_2| < 1$. If follows that

$$a_4 \cdot \det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix} = 0. \tag{35}$$

Similarly, (30), (33) imply

$$a_4 \cdot \det \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix} = 0 \tag{36}$$

and (31), (34) imply that

$$a_4 \cdot \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0. \tag{37}$$

Note that $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ has rank 3 and $(a_1, a_2, a_3)^t$ is not zero. Then $\det \begin{bmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{bmatrix}$, $\det \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$ and $\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ cannot be all zero. This together with (35)-(37) implies that $a_4 = 0$. Similar argument will yield $a_1 = a_2 = a_3 = 0$. This is a contradiction and the claim is thus proved.

We now assume that $a_{j_0} \neq 0$ for some $1 \leq j_0 \leq n+1$ and all other $a_j = 0$. It follows from $\mathbf{U} \in U(n,1)$ that $|a_{j_0}| = 1$. Write $a_{j_0} = e^{\sqrt{-1}\theta}$ for $\theta \in [0, 2\pi)$. Again by $\mathbf{U} \in U(n,1)$ we have the j_0-th row of $\mathbf{U}$ is $(e^{\sqrt{-1}\theta}, 0, \cdots, 0)$. Interchange the first and j_0-th row of $\mathbf{U}$ and still denote the new matrix by $\mathbf{U}$. Hence one has

$$\mathbf{U} = \begin{bmatrix} e^{\sqrt{-1}\theta} & \mathbf{0}_{1\times n} \\ \mathbf{0}^t_{1\times n} & \mathbf{V} \end{bmatrix} \tag{38}$$

and

$$\mathbf{U} \cdot \mathrm{diag}(\lambda, \lambda_2, \cdots, \lambda_{n+1}) \cdot \mathbf{U}^t = c \cdot \mathrm{diag}(\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_n}), \tag{39}$$

where $\{j_1, \cdots, j_n\}$ is a permutation of $\{1, \cdots, n+1\} \setminus \{j_0\}$. It follows from (38), (39) that

$$e^{2\sqrt{-1}\theta}\lambda = c\lambda_{j_0} \tag{40}$$

and

$$\mathbf{V} \cdot \mathrm{diag}(\lambda_2, \cdots, \lambda_{n+1}) \cdot \mathbf{V}^t = c \cdot \mathrm{diag}(\lambda_{j_1}, \cdots, \lambda_{j_n}). \tag{41}$$

Recall that $|\lambda| < |\lambda_1| \leq \cdots \leq |\lambda_{n+1}|$. (40) implies that $|c| < 1$. Note $\det(\mathbf{V}) = 1$ as $\det(\mathbf{U}) = 1$. Therefore (41) implies

$$|c| = \prod_{j=2}^{n+1} |\lambda_j| / \prod_{k=1}^{n} |\lambda_{j_k}| \geq 1.$$

This is a contradiction and thus the proposition is proved. □

3.5 Degree estimates

We will establish a optimal degree estimate result for rational holomorphic isometries from $\mathbb{B}^n$ to D^{IV}_m.

Definition 3.12. *Let F be a rational map from $\mathbb{C}^n$ into $\mathbb{C}^m$. We write*

$$F = \frac{(P_1, \cdots, P_m)}{R}$$

where $P_j, j = 1, \cdots, m$ and R are holomorphic polynomials, their greatest common divisor $(P_1, \cdots, P_m, R) = 1$. The degree of F, denoted by $\deg F$, is defined to be

$$\deg F := max\{\deg(P_j), j = 1, \cdots, m, \deg R\}.$$

Theorem 3.13. *Assume $m \geq n+1$. Let $F : \mathbb{B}^n \to D_m^{IV}$ be a rational holomorphic isometric embedding satisfying $F(0) = 0$ and*

$$F^* \omega_{D_m^{IV}} = \omega_{\mathbb{B}^n}.$$

Then deg$(F) \leq 2$. Moreover, if $n \geq 2$, then F is either a homogeneous linear polynomial map or deg$(F) = 2$.

Proof. Write $F = (f_1, \cdots, f_m)$ and $h = \frac{\sum_{i=1}^m f_i^2}{2}$. It follows from the isometry assumption that

$$\sum_{j=1}^m |f_j(z)|^2 = \sum_{j=1}^n |z_j|^2 + |h(z)|^2. \tag{42}$$

By Calabi [C] and D'Angelo's theorem [D2], there exists a unitary matrix $\mathbf{U} = (u_{ij}) \in M(m, m; \mathbb{C})$ such that

$$\left(\frac{1}{2} \sum_{j=1}^m f_j^2(z), z_1, \cdots, z_n, 0, \cdots, 0 \right) \cdot \mathbf{U} = (f_1(z), \cdots, f_m(z)). \tag{43}$$

Equation (43) reads

$$f_j(z) = u_{1j} h(z) + \sum_{i=1}^n u_{i+1,j} z_i \tag{44}$$

for all $1 \leq j \leq m$. Take the sum of square of the above equations for all j, we conclude that

$$\begin{aligned} 2h &= \left(\sum_{j=1}^m u_{1j}^2 \right) h^2 + 2h \sum_{j=1}^m \left(u_{1j} \sum_{i=1}^n u_{i+1,j} z_i \right) + \sum_{j=1}^m \left(\sum_{i=1}^n u_{i+1,j} z_i \right)^2 \\ &= \left(\sum_{j=1}^m u_{1j}^2 \right) h^2 + 2h \sum_{j=1}^m \left(u_{1j} \sum_{i=1}^n u_{i+1,j} z_i \right) + \sum_{i=1}^n \left(\sum_{j=1}^m u_{i+1,j}^2 \right) z_i^2 + 2 \sum_{i \neq i'} \left(\sum_{j=1}^m u_{i+1,j} u_{i'+1,j} \right) z_i z_i'. \end{aligned} \tag{45}$$

This is a quadratic equation

$$ah^2(z) + (p(z) - 2)h(z) + q(z) = 0 \tag{46}$$

where $a = \sum_{j=1}^{n+1} u_{1j}^2$ and $p(z)$ and $q(z)$ are homogeneous polynomials in z of degree 1 and 2 respectively. Assume that h is rational. When $a \neq 0$, there is a contradiction to Vieta's theorem. When $a = 0$, then h must be a rational function of degree 2 in z or identically 0. If $h \equiv 0$, then f_j are homogeneous linear polynomials for all j. Otherwise, F is a rational function in z of degree at most 2 by equation (44).

Now assume $h \not\equiv 0$. Then h can be solved from (45) or (46) with $h(z) = \frac{q(z)}{2-p(z)}$ for $q \not\equiv 0$. We will show that $\deg(F) = 2$. If $p \equiv 0$, it is trivially true. Otherwise, by (44),

$$f_j(z) = \frac{-u_{1j}q(z) + p(z)\left(\sum_{i=1}^n u_{i+1,j}z_i\right) - 2\left(\sum_{i=1}^n u_{i+1,j}z_i\right)}{p(z)-2},$$

denoted by $\frac{N_j(z)}{p(z)-2}$ for $1 \leq j \leq m$. We claim that there exists at least one j_0 such that $\deg(N_{j_0}(z)) = 2$. Therefore $\deg(f_{j_0}(z)) = 2$ as q cannot be divided by $p-2$. Suppose $\deg(N_j(z)) = 1$ for all $1 \leq j \leq m$. This is equivalent to

$$u_{1j}q(z) = p(z)\left(\sum_{i=1}^n u_{i+1,j}z_i\right). \tag{47}$$

Assume $n \geq 2$. Then the matrix $\begin{bmatrix} u_{21}, \cdots, u_{2m} \\ \cdots, \cdots, \cdots \\ u_{(n+1)1}, \cdots, u_{(n+1)m} \end{bmatrix}$ is of rank equal to $n \geq 2$. Therefore, there exists $1 \leq j_1 < j_2 \leq m$ such that $\begin{bmatrix} u_{2j_1} \\ \cdots \\ u_{(n+1)j_1} \end{bmatrix}$ and $\begin{bmatrix} u_{2j_2} \\ \cdots \\ u_{(n+1)j_2} \end{bmatrix}$ are linearly independent and thus

$$\sum_{i=1}^n u_{i+1,j_1}z_i \not\equiv 0, \sum_{i=1}^n u_{i+1,j_2}z_i \not\equiv 0, \left(\sum_{i=1}^n u_{i+1,j_1}z_i, \sum_{i=1}^n u_{i+1,j_2}z_i\right) = 1.$$

It follows from (47) that $u_{1j_1} \neq 0, u_{1j_2} \neq 0$ and moreover,

$$q(z) = \frac{p(z)\left(\sum_{i=1}^n u_{i+1,j_1}z_i\right)}{u_{1j_1}} = \frac{p(z)\left(\sum_{i=1}^n u_{i+1,j_2}z_i\right)}{u_{1j_2}}.$$

This contradicts to the linear independence of $\begin{bmatrix} u_{2j_1} \\ \cdots \\ u_{(n+1)j_1} \end{bmatrix}$ and $\begin{bmatrix} u_{2j_2} \\ \cdots \\ u_{(n+1)j_2} \end{bmatrix}$.

Therefore we proved that either $\deg(F) = 2$ or F is a homogeneous linear polynomial map when $n \geq 2$. $\square$

We have more precise result for $m < 2n$.

Theorem 3.14. *Assume $n+1 \le m < 2n, n \ge 2$. Let $F : \mathbb{B}^n \to D_m^{IV}$ be a rational holomorphic isometric embedding satisfying $F(0) = 0$ and*

$$F^*\omega_{D_m^{IV}} = \omega_{\mathbb{B}^n}.$$

Then deg$(F) = 2$.

Proof. By Theorem 3.13, we have $\deg(F) = 1$ or 2. Hence we just need to show $\deg(F)$ cannot be 1. We will prove by seeking a contradiction. Suppose $\deg(F) = 1$. By the argument in Theorem 3.13, each $f_i, 1 \le i \le m$, is a homogeneous linear polynomial in z. Then $\sum_{j=1}^m f_j^2(z)$ will be a homogeneous quadratic polynomial in z. By collecting terms of degree 4 on both sides of (42), we have $\sum_{j=1}^m f_j^2(z) = 0$. Equation (43) is then reduced to,

$$(z_1, \cdots, z_n, 0, \cdots, 0)\,\mathbf{V} = (f_1(z), \cdots, f_m(z))\,. \tag{48}$$

Here $\mathbf{V} = \begin{pmatrix} \mathbf{v_1} \\ \dots \\ \mathbf{v_m} \end{pmatrix}$ is an $m \times m$ unitary matrix, $\mathbf{v_i}$ is an $m-$dimensional row vector, $1 \le i \le m$. We rewrite (48) as

$$(z_1, ..., z_n) \begin{pmatrix} \mathbf{v}_1 \\ \dots \\ \mathbf{v}_n \end{pmatrix} = (f_1, ..., f_m)\,. \tag{49}$$

The fact that $\sum_{j=1}^m f_j^2(z) = 0$, implies

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0, \text{ for all } 1 \le i, j \le n. \tag{50}$$

As $\mathbf{V}$ is an unitary matrix, we have,

$$\mathbf{v}_i \cdot \overline{\mathbf{v}}_j = 0, \text{ for all } 1 \le i \ne j \le n; \tag{51}$$

$$\mathbf{v}_i \cdot \overline{\mathbf{v}}_i = 1, 1 \le i \le n. \tag{52}$$

It follows from equations (50), (51) and (52) that

$$\mathrm{Re}\mathbf{v}_i \cdot \mathrm{Im}\mathbf{v}_i = 0, \mathrm{Re}\mathbf{v}_i \cdot \mathrm{Re}\mathbf{v}_i = \mathrm{Im}\mathbf{v}_i \cdot \mathrm{Im}\mathbf{v}_i = \frac{1}{2}, 1 \le i \le n. \tag{53}$$

$$\mathrm{Re}\mathbf{v}_i \cdot \mathrm{Im}\mathbf{v}_j = 0 \text{ for all } 1 \le i, j \le n. \tag{54}$$

We thus get a collection of $2n$ mutually orthogonal nonzero real vectors $\{\mathrm{Re}\mathbf{v}_i, \mathrm{Im}\mathbf{v}_i\}_{i=1}^n$ in $\mathbb{C}^m$. This contradicts to the assumption that $m < 2n$. We thus establishes Theorem 3.14. □

Remark 3.15. *We remark that the assumption $m < 2n$ is optimal in Theorem 3.14. Indeed, when $m = 2n$, we have a linear map $F : \mathbb{B}^n \to D_{2n}^{IV}$:*

$$F(z) = (\frac{\sqrt{2}}{2}z_1, \frac{\sqrt{-2}}{2}z_1, ..., \frac{\sqrt{2}}{2}z_n, \frac{\sqrt{-2}}{2}z_n).$$

Furthermore, we have the following rigidity result for holomorphic rational isometric map of degree one.

Proposition 3.16. *Assume* $m \geq 2n$ *and* $n \geq 2$. *Let* $F : \mathbb{B}^n \to D_m^{IV}$ *be a rational holomorphic isometric embedding satisfying* $F(0) = 0$ *and*

$$F^*\omega_{D_m^{IV}} = \omega_{\mathbb{B}^n}.$$

Assume that $\deg(F) = 1$. *Then* F *is a totally geodesic embedding that is isotropically equivalent to*

$$(\frac{\sqrt{2}}{2}z_1, i\frac{\sqrt{2}}{2}z_1, ..., \frac{\sqrt{2}}{2}z_n, i\frac{\sqrt{2}}{2}z_n, 0, ..., 0). \tag{55}$$

Proof. Recall from the proof of Theorem 3.13 and 3.14, we have if $\deg(F) = 1$, then F is a homogeneous linear map. More precisely, there is an $m \times m$ unitary matrix $V = \begin{pmatrix} \mathbf{v}_1 \\ ... \\ \mathbf{v}_m \end{pmatrix}$ such that equations (48)-(54) hold.

We write $\mathbf{a}_i = \mathrm{Re}\mathbf{v}_i, \mathbf{b}_i = \mathrm{Im}\mathbf{v}_i, 1 \leq i \leq n$, and write the $2n \times m$ matrix,

$$\mathbf{C} = \sqrt{2}\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_1 \\ ... \\ ... \\ \mathbf{a}_n \\ \mathbf{b}_n \end{pmatrix}.$$

As a consequence of (53), (54), we have

$$\mathbf{C}\mathbf{C}^t = \mathbf{I}_n.$$

We extend $\{\sqrt{2}\mathbf{a}_j, \sqrt{2}\mathbf{b}_j\}_{j=1}^n$ to an orthonormal basis $\{\sqrt{2}\mathbf{a}_1, \sqrt{2}\mathbf{b}_1, ..., \sqrt{2}\mathbf{a}_n, \sqrt{2}\mathbf{b}_n, \mathbf{c}_{2n+1}, ..., \mathbf{c}_m\}$ of $\mathbb{R}^m$. We then set $\widetilde{\mathbf{C}}$ to be the $m \times m$ matrix,

$$\widetilde{\mathbf{C}}^t = \begin{pmatrix} \sqrt{2}\mathbf{a}_1 \\ \sqrt{2}\mathbf{b}_1 \\ ... \\ ... \\ \sqrt{2}\mathbf{a}_n \\ \sqrt{2}\mathbf{b}_n \\ \mathbf{c}_{2n+1} \\ ... \\ \mathbf{c}_m \end{pmatrix},$$

then

$$\widetilde{\mathbf{C}}\widetilde{\mathbf{C}}^t = \mathbf{I}_m. \tag{56}$$

We now define

$$\widetilde{F} = F\widetilde{\mathbf{C}}^t.$$

Then $\widetilde{F}$ is orthogonal equivalent to F. Moreover,

$$\widetilde{F} = (z_1, ..., z_n) \begin{pmatrix} \mathbf{v}_1 \\ ... \\ \mathbf{v}_n \end{pmatrix} \widetilde{\mathbf{C}}^t = (z_1, ..., z_n) \begin{pmatrix} \mathbf{a}_1 + \sqrt{-1}\mathbf{b}_1 \\ ... \\ \mathbf{a}_n + \sqrt{-1}\mathbf{b}_n \end{pmatrix} \widetilde{\mathbf{C}}^t. \tag{57}$$

It then follows from (56) that

$$\widetilde{F} = \left(\frac{\sqrt{2}}{2} z_1, i\frac{\sqrt{2}}{2} z_1, ..., \frac{\sqrt{2}}{2} z_n, i\frac{\sqrt{2}}{2} z_n, 0, ..., 0 \right).$$

□

Remark 3.17. *$n = 1$ is a special case ($n + 1 = 2n$). Let F be a holomorphic isometry from $\Delta \subset \mathbb{C}$ to D_2^{IV} satisfying $F^*\omega_{D_2^{IV}} = \omega_\Delta$. It follows from Theorem* **??** *and Remark 3.15 that F is either equivalent to the totally geodesic embedding $(\sqrt{2}z/2, \sqrt{-2}z/2)$ or equivalent to the non-totally geodesic embedding $I_{1,0}$. Note that D_2^{IV} is biholomorphic to bidisc $\Delta^2 \subset \mathbb{C}^2$. In the Euclidean coordinate of Δ^2, the first map is $z \to (z, 0)$ and $I_{1,0}$ is the square root embedding constructed by Mok [M5]. This classification result from Δ to Δ^2 was obtained earlier by Ng ([Ng1] Theorem 7.1).*

We end the paper by presenting an elementary proof of Theorem 2.1 (ii) of Mok for Type IV domains.

Proposition 3.18. *Let $U \subset \mathbb{B}^m$ be a connected open set and $F : U \to D_{n+1}^{IV}$ be a holomorphic isometry satisfying*

$$F^*\omega_{D_{n+1}^{IV}} = \omega_{\mathbb{B}^m}.$$

Then $m \leq n$.

Proof. Write $F = (f_1, \cdots, f_{n+1})$. Without loss of generality, assume $0 \in U$ and $F(0) = 0$. Then the isometry assumption is equivalent to $\sum_{j=1}^m |z_j|^2 + |\sum_{j=1}^{n+1} f_j^2(z)|^2 = \sum_{j=1}^{n+1} |f_j(z)|^2$. It follows by Calabi's theorem [C] or D'Angelo's theorem [D2] that the rank of $\sum_{j=1}^m |z_j|^2 + |\sum_{j=1}^{n+1} f_j^2(z)|^2$ must be equal to the rank of $\sum_{j=1}^{n+1} |f_j(z)|^2$. The former is at least m and the latter is at most $n+1$. Thus $m \leq n+1$. Suppose $m = n+1$. Then $\sum_{j=1}^m f_j^2(z)$ and all $f_j(z)$ for $1 \leq j \leq m$ can be written as linear combinations of $\{z_1, \ldots, z_m\}$. This implies that $\sum_{j=1}^m f_j^2(z) \equiv 0$. Applying Calabi's theorem or D'Angelo's theorem again to $\sum_{j=1}^m |z_j|^2 = \sum_{j=1}^m |f_j(z)|^2$, we know that there exists an $m \times m$ unitary matrix $\mathbf{U}$ such that $(f_1, \cdots, f_m) = (z_1, \cdots, z_m) \cdot \mathbf{U}$. Therefore $0 \equiv \sum_{j=1}^m f_j^2(z) = (z_1, \cdots, z_m) \cdot \mathbf{U} \cdot \mathbf{U}^{\mathbf{t}} \cdot (z_1, \cdots, z_m)^t$. This is impossible. Hence we showed $m \leq n$. □

4 Polynomial non-totally geodesic holomorphic isometries

4.1 Singularities of holomorphic isometries

The rational holomorphic isometries R_*^* given in previous sections from the unit ball $\mathbb{B}^n$ into an irreducible classic symmetric domain Ω are not totally geodesic and only produce singularities at one single point on the boundary $\partial\mathbb{B}^n$. When $n \geq 2$, one can easily avoid passing through this point by slicing $\mathbb{B}^n$ with a complex linear hyperplane. Therefore, one obtains holomorphic polynomial isometries from $\mathbb{B}^{n-1}$ into Ω. In particular, this answers the question raised by Mok in [M4] (Question 5.2.2) in the negative in general while it may still hold in the maximal dimensional case. Indeed, we will show in Section 4 that Mok's conjecture holds for holomorphic isometries from the unit ball into the irreducible bounded symmetric domain of type IV in the maximal dimensional case.

Theorem 4.1. *There exist non-totally geodesic holomorphic isometries from the unit ball $\mathbb{B}^m$ into the irreducible bounded symmetric domain that can be extended holomorphically to $\mathbb{C}^m$.*

Proof. Indeed, there are polynomial holomorphic isometries. They are given by

$$\begin{pmatrix} 0 & z_2 & \dots & z_q \\ w_2 & -w_2 z_2 & \dots & -w_2 z_p \\ \dots & \dots & \dots & \dots \\ w_p & -w_p z_2 & \dots & -w_p z_q \end{pmatrix} : \mathbb{B}^{p+q-2} \to D^I_{p,q} \text{ for } p \geq q \geq 2;$$

$$\begin{pmatrix} 0 & 0 & z_3 & z_4 & \dots & z_n \\ 0 & 0 & w_3 & w_4 & \dots & w_n \\ -z_3 & -w_3 & 0 & w_3 z_4 - z_3 w_4 & \dots & z_n w_3 - z_3 w_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -z_{n-1} & \dots & \dots & \dots & 0 & z_n w_{n-1} - z_{n-1} w_n \\ -z_n & -w_n & \dots & \dots & \dots & 0 \end{pmatrix} : \mathbb{B}^{2n-4} \to D^{II}_n \text{ for } n \geq 4;$$

$$\begin{pmatrix} 0 & \frac{z_2}{\sqrt{2}} & \dots & \frac{z_n}{\sqrt{2}} \\ \frac{z_2}{\sqrt{2}} & -\frac{z_2^2}{2} & \dots & -\frac{z_2 z_n}{2} \\ \dots & \dots & \dots & \dots \\ \frac{z_n}{\sqrt{2}} & -\frac{z_2 z_n}{2} & \dots & -\frac{z_2^2}{2} \end{pmatrix} : \mathbb{B}^{n-1} \to D^{III}_n \text{ for } n \geq 2;$$

$$\left(z_1, \cdots, z_{n-1}, -\frac{\sqrt{2}}{4} \sum_{i=1}^{n-1} z_i^2, \frac{\sqrt{-2}}{4} \sum_{i=1}^{n-1} z_i^2 \right) : \mathbb{B}^{n-1} \to D^{IV}_{n+1} \text{ for } n \geq 2.$$

□

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