

ON THE EMBEDDABILITY OF REAL HYPERSURFACES INTO HYPERQUADRICS

ILYA KOSSOVSKIY AND MING XIAO

ABSTRACT. A well known result of Forstnerić [18] states that most real-analytic strictly pseudoconvex hypersurfaces in complex space are not holomorphically embeddable into spheres of higher dimension. A more recent result by Forstnerić [19] states even more: most real-analytic hypersurfaces do not admit a holomorphic embedding even into a merely algebraic hypersurface of higher dimension, in particular, a hyperquadric. Explicit examples of real-analytic hypersurfaces non-embeddable into hyperquadrics were obtained by Zaitsev [38]. In contrast, the classical theorem of Webster [37] asserts that every *real-algebraic* Levi-nondegenerate hypersurface admits a transverse holomorphic embedding into a nondegenerate real hyperquadric in complex space.

In this paper, we provide *effective* results on the non-embeddability of real-analytic hypersurfaces into a hyperquadric. We show that, for any $N > n \geq 1$, the defining functions $\varphi(z, \bar{z}, u)$ of all real-analytic hypersurfaces $M = \{v = \varphi(z, \bar{z}, u)\} \subset \mathbb{C}^{n+1}$ containing Levi-nondegenerate points and locally transversally holomorphically embeddable into some hyperquadric $\mathcal{Q} \subset \mathbb{C}^{N+1}$ satisfy an *universal* algebraic partial differential equation $D(\varphi) = 0$, where the algebraic-differential operator $D = D(n, N)$ depends on n, N only. To the best of our knowledge, this is the first effective result characterizing real-analytic hypersurfaces embeddable into a hyperquadric of higher dimension. As an application, we show that for every n, N as above there exists $\mu = \mu(n, N)$ such that a Zariski generic real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ of degree $\geq \mu$ is not transversally holomorphically embeddable into any hyperquadric $\mathcal{Q} \subset \mathbb{C}^{N+1}$. We also provide an explicit upper bound for μ in terms of n, N . To the best of our knowledge, this gives the first effective lower bound for the CR-complexity of a Zariski generic real-algebraic hypersurface in complex space of a fixed degree.

CONTENTS

1.	Introduction	2
2.	Preliminaries	5
2.1.	Transversality of CR-embeddings	5
2.2.	Description of the principal method	6
3.	Real-analytic hypersurfaces in \mathbb{C}^2 embeddable into hyperquadrics in \mathbb{C}^3	7
4.	The high dimensional case	12
5.	Appendix I	21
6.	Appendix II	23
	References	26

1. INTRODUCTION

Let $M \subset \mathbb{C}^{n+1}$, $n \geq 1$ be a real-analytic Levi-nondegenerate hypersurface. The celebrated theory due to Chern and Moser [7] (see also Cartan [5]) asserts that only a very rare such M admits a local biholomorphic mapping into a nondegenerate real hyperquadric

$$\mathcal{Q} = \{[\xi_0, \dots, \xi_{n+1}] \in \mathbb{C}\mathbb{P}^{n+1} : |\xi_0|^2 + \dots + |\xi_k|^2 - |\xi_{k+1}|^2 - \dots - |\xi_{n+1}|^2 = 0\}.$$

Moreover, Chern and Moser show that the existence of the desired biholomorphic mapping into a hyperquadric is equivalent to vanishing of a special *CR-curvature* of a real hypersurface M .

A natural problem to pursue in view of the Chern-Moser theory is the possibility to construct a local holomorphic embedding $F : (M, p) \mapsto (\mathcal{Q}, p')$ of a real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 1$ into a hyperquadric $\mathcal{Q} \subset \mathbb{C}^{N+1}$ of *higher dimension*. Here by a holomorphic embedding F of $M \subset \mathbb{C}^n$ into $M' \subset \mathbb{C}^N$, we mean a holomorphic embedding of an open neighborhood U of M in \mathbb{C}^n into a neighborhood U' of M' in \mathbb{C}^N , sending M into M' . One usually presumes certain nondegeneracy conditions for the mapping F , such as *transversality* (the latter means that $dF(\mathbb{C}^{n+1})|_p \not\subset T_{p'}\mathcal{Q}$).

The existence of a transversal holomorphic embedding into a hyperquadric can be viewed as a *finite CR-complexity* of a real hypersurface (see, e.g., Ebenfelt and Shroff [14]). The latter number is the minimal possible difference $N - n$ between the CR-dimensions of the target hyperquadric and the source real hypersurface. An alternative approach to complexity in CR-geometry is due to the school of D'Angelo, see, e.g., [8, 9, 10]. A strong motivation for studying the embedding problem is the celebrated theorem of Webster [37] which states that *every* real-algebraic Levi-nondegenerate hypersurface admits a transverse holomorphic embedding into a nondegenerate real hyperquadric in complex space. Thus, every algebraic Levi-nondegenerate hypersurface has a finite CR-complexity.

Since the work of Webster, a large number of publications have been dedicated to studying holomorphic embeddings of real hypersurfaces into hyperquadrics. However, despite of the extensive research in this direction, the following problem remains widely open:

Problem 1: Characterize the embeddability of a real hypersurface $M \subset \mathbb{C}^{n+1}$ into a hyperquadric $\mathcal{Q}^{2N+1} \subset \mathbb{C}^{N+1}$. More precisely, find a necessary and sufficient condition for M to admit a transversal holomorphic embedding into some $\mathcal{Q}^{2N+1} \subset \mathbb{C}^{N+1}$.

We emphasize in connection with Problem 1 that not every Levi-nondegenerate real-analytic hypersurface can be transversally holomorphically embedded into a hyperquadric. Indeed, a well known result of Forstnerić [18] (see also Faran [17]) states that most real-analytic strictly pseudoconvex hypersurfaces are not holomorphically embeddable into spheres of higher dimension. A more recent result by Forstnerić [19] states even more: most real-analytic hypersurfaces do not admit a holomorphic embedding even into a merely algebraic hypersurface of higher dimension. Importantly, both cited theorems are proved by showing that the set of embeddable hypersurfaces is a set of first Baire category. An important step towards understanding the embeddability property was done by Zaitsev [38], who obtained explicit examples of Levi-nondegenerate real-analytic hypersurfaces that are not transversally holomorphically embeddable into any hyperquadrics. We also mention the recent work

of Huang and Zaitsev [21] and Huang and Zhang [23], where the authors construct concrete algebraic Levi-nondegenerate hypersurfaces with positive signature which can not be holomorphically embedded into a hyperquadric with the same signature of any dimension.

However, the cited results still leave open the question on an *effective* characterization of the set of real-analytic Levi-nondegenerate hypersurfaces in \mathbb{C}^{n+1} , admitting a local transversal holomorphic embedding into a hyperquadric in \mathbb{C}^{N+1} . That is, we are searching for a more constructive characterization of the set of embeddable hypersurfaces than the one in [19]. Theorem 1 below provides such a characterization for *any* fixed n, N . Namely, we show that for any fixed n, N with $n \geq 1, n < N$ the set of embeddable hypersurfaces $M = \{v = \varphi(z, \bar{z}, u)\} \subset \mathbb{C}^{n+1}$ satisfies an *universal* algebraic partial differential equation

$$D(\varphi) = 0,$$

where the differential-algebraic operator $D = D(n, N)$ depends on n, N only. Thus, the defining functions of embeddable hypersurfaces form a subset of a *differential-algebraic set*. Following the method of the present paper, each differential-algebraic operator $D(n, N)$ can be *effectively* computed (see Remark 1.3 below), as well as an effective bound for its degree can be obtained immediately (see Appendix I).

The other question addressed in the paper is connected to Webster's embedding theorem mentioned above. Motivated by embedding theorems in various geometries (such as Whitney embedding theorem in differential topology and Remmert theorem in the Stein space theory) it is natural, in view of Webster's theorem, to ask the following.

Problem 2. *Is there a uniform embedding dimension N which only depends on n such that all Levi-nondegenerate real-algebraic hypersurfaces $M \subset \mathbb{C}^{n+1}$ can be transversally holomorphically embedded into a hyperquadric of suitable signature in \mathbb{C}^{N+1} ? In other words, is there a uniform upper bound for the CR-complexity of all Levi-nondegenerate real-algebraic hypersurfaces $M \subset \mathbb{C}^{n+1}$?*

A closely related problem is as follows.

Problem 3. *Provide an effective bound for the CR-complexity of a (generic) real-algebraic hypersurface $M \subset \mathbb{C}^{n+1}$ of a fixed degree k in terms of n and k .*

By applying Theorem 1, we give a negative answer to Problem 2 (see Theorem 2). Moreover, Theorem 2 gives an *explicit* constant $\mu = \mu(n, N)$ such that a Zariski-generic algebraic hypersurface of any fixed degree $k \geq \mu$ is not transversally holomorphically embeddable into any hyperquadric in \mathbb{C}^{N+1} , thus providing a solution for Problem 3.

We now formulate our results in detail. We first recall the concept of a differential-algebraic operator. Let $n, l \geq 1$ be integers and P be a polynomial defined on the space $J^l(\mathbb{C}^n, \mathbb{C})$ of jets of maps from \mathbb{C}^n to \mathbb{C} . Then P uniquely defines a *differential-algebraic operator* $\mathcal{D} = \mathcal{D}(P)$, which is the differential operator acting on an analytic function $\rho : U \mapsto \mathbb{C}$ by

$$\mathcal{D}(\rho) := P(j^l \rho)$$

(here $U \subset \mathbb{C}^n$ is a domain). The integer l is called its *order*. We call a differential-algebraic operator *shift-invariant*, if it is invariant under shifts in $j^0\rho$ (that is, $\mathcal{D}(\rho)$ does not depend on z_1, \dots, z_n, ρ explicitly and depends on derivatives of ρ of order at least 1).

Theorem 1. *For any integers $N > n \geq 1$, there exists a universal non-zero shift-invariant differential-algebraic operator $D = D(n, N)$ such that the following holds. If a real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ with a defining equation*

$$v = \varphi(z, \bar{z}, u)$$

contains at least one Levi-nondegenerate point and admits a local transverse holomorphic embedding into a hyperquadric $\mathcal{Q}^{2N+1} \subset \mathbb{C}^{N+1}$ near some point $p_0 \in M$, then

$$D(\varphi) \equiv 0.$$

To the best of our knowledge, Theorem 1 gives first effective results characterizing real-analytic hypersurfaces embeddable into a hyperquadric of higher dimension. We shall also note that a weaker version of the assertion of Theorem 1 (a differential-algebraic relation for jets of the Segre varieties $\{Q_p\}_{p \in Q_0}$) was proved earlier by Zaitsev in [38].

We now give a series of important remarks.

Remark 1.1. We remark that, according to [12], we may drop the transversality requirement in Theorem 1 in the case $N < 2n$, as the latter holds automatically.

Remark 1.2. In Section 3 we show that in the case $n = 1, N = 2$ the order of the differential-algebraic operator in Theorem 1 equals to 18. An explicit bound for the order of $D(n, N)$ in the general case can be verified from the Appendix I.

Remark 1.3. The proof of Theorem 1 given in Sections 3 and 4 in fact provides an algorithm for finding the differential-algebraic operator $D(\varphi)$ *explicitly* for any fixed dimensions n, N . More precisely, the proof shows that $D(\varphi)$ is a finite product of explicit determinants involving the *PDE defining function* $\{\Phi_{ij}\}_{i,j=1}^n$ of a real hypersurface and its derivatives (see Section 2.2 for details of the concept). In Sections 3 and 4 we also provide an algorithm for recalculating all the relevant derivatives of the PDE defining function $\{\Phi_{ij}\}_{i,j=1}^n$ in terms of derivatives of the initial defining function $\varphi(z, \bar{z}, u)$.

By using Theorem 1, we obtain the following effective bound for embeddability of real-algebraic hypersurfaces into hyperquadrics.

Theorem 2. *For any integers $N > n \geq 1$, there exists $\mu = \mu(n, N)$ such that a Zariski generic real-algebraic hypersurface $M \subset \mathbb{C}^{n+1}$ of any degree $k \geq \mu$ is not transversally holomorphically embeddable into a hyperquadric $\mathcal{Q}^{2N+1} \subset \mathbb{C}^{N+1}$. An explicit bound for $\mu(n, N)$ is given in Theorem 5 below (see Appendix I).*

Thus, Theorem 2 provides a solution for Problem 2. As was mentioned above, it also gives the first effective lower bound for the CR-complexity of a Zariski-generic real-algebraic hypersurface in complex space of a fixed degree, thus giving a solution to Problem 3.

Remark 1.4. Note that in the special case $n = 1$, $N = 2$ we may take $\mu = 18$ for the degree bound, as shown in Section 3.

The main tool of the paper is the recent dynamical technique in CR-geometry, which shall be addressed as the *method of associated differential equations* in the non-singular setting (see Sukhov [33, 34]), and the *CR – DS technique* in the singular one (see the work of Lamel, Shafikov and the first author [26, 27, 24]). An overview of the technique is given in Section 2.2.

Acknowledgements.

The authors acknowledge AIM for holding a workshop on Cauchy-Riemann equations in several variables in June 2014, where they started the work. The authors are grateful to Valeri Beloshapka, John D’Angelo, Peter Ebenfelt, and Xiaojun Huang for their interest and helpful discussions, and are particularly grateful to Dmitri Zaitsev for his very valuable comments on the initial version of the manuscript, which allowed to significantly strengthen the main results of the paper.

The first author is supported by the Austrian Science Fund (FWF).

2. PRELIMINARIES

2.1. Transversality of CR-embeddings.

We first recall the notion of transversality. If U is an open subset of \mathbb{C}^{n+1} , H a holomorphic mapping $U \mapsto \mathbb{C}^{N+1}$, and M' a real hypersurface through a point $H(p)$ for some $p \in U$, then H is said to be transversal to M' at $H(p)$ if

$$T_{H(p)}M' + dH(T_p\mathbb{C}^{n+1}) = T_{H(p)}\mathbb{C}^{N+1},$$

where $T_p\mathbb{C}^{n+1}$ and $T_{H(p)}M'$ denote the real tangent spaces of \mathbb{C}^{n+1} and M' at p and $H(p)$, respectively. We here mention that in our setting where there is a real hypersurface $M \subset U$ such that $H(M) \subset M'$, the notion of transversality of a mapping to a hypersurface coincides with that of CR transversality (cf. [1]). We also recall that H is called CR transversal if $dF(\mathbb{C}T_pM)$ is not contained in $\mathcal{V}'_{F(p)} + \overline{\mathcal{V}'_{F(p)}}$, where \mathcal{V}' is the CR bundle of M' . Note that a CR mapping is CR transversal at $p \in M$ is equivalent to the nonvanishing of the derivative of its normal component at p along the normal direction (cf. [1]).

2.2. Description of the principal method. It was observed by Cartan [5, 6] and Segre [32] (see also Webster [37]) that the geometry of a real hypersurface in \mathbb{C}^2 parallels that of a second order ODE

$$w'' = \Phi(z, w, w'). \quad (2.1)$$

More generally, the geometry of a real hypersurface in \mathbb{C}^{n+1} , $n \geq 2$ parallels that of a complete second order system of PDEs

$$w_{z_k z_l} = \Phi_{kl}(z_1, \dots, z_n, w, w_{z_1}, \dots, w_{z_n}), \quad k, l = 1, \dots, n. \quad (2.2)$$

Moreover, this parallel becomes algorithmic by using the Segre family of a real hypersurface. With any real-analytic Levi-nondegenerate hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 1$ one can uniquely associate an ODE (2.1) ($n = 1$) or a PDE system (2.2) ($n \geq 2$). The Segre family of M plays a role of a mediator between the hypersurface and the associated differential equations. A modern clear exposition of the method was given in the work [33, 34] of Sukhov.

The associated differential equations procedure is particularly clear in the case of a Levi-nondegenerate hypersurface in \mathbb{C}^2 . In this case the Segre family is a 2-parameter anti-holomorphic family of pairwise transverse holomorphic curves. It immediately follows then from the main ODE theorem that there exists a unique ODE (2.1), for which the Segre varieties are precisely the graphs of solutions. This ODE is called *the associated ODE*.

Let us provide some details in the general case. We denote the coordinates in \mathbb{C}^{n+1} by $(z, w) = (z_1, \dots, z_n, w)$. Let $M \subset \mathbb{C}^{n+1}$ be a smooth real-analytic hypersurface, passing through the origin, and choose a small neighborhood U of the origin. In this case we associate a complete second order system of holomorphic PDEs to M , which is uniquely determined by the condition that the differential equations are satisfied by all the graphing functions $h(z, \zeta) = w(z)$ of the Segre family $\{Q_\zeta\}_{\zeta \in U}$ of M in a neighbourhood of the origin. To be more explicit we consider the so-called *complex defining equation* (see, e.g., [1]) $w = \rho(z, \bar{z}, \bar{w})$ of M near the origin, which one obtains by substituting $u = \frac{1}{2}(w + \bar{w})$, $v = \frac{1}{2i}(w - \bar{w})$ into the real defining equation and applying the holomorphic implicit function theorem. The Segre variety Q_p of a point $p = (a, b) \in U$, $a \in \mathbb{C}^n$, $b \in \mathbb{C}$ is now given as the graph

$$w(z) = \rho(z, \bar{a}, \bar{b}). \quad (2.3)$$

Differentiating (2.3) once with respect to all variables, we obtain

$$w_{z_j} = \rho_{z_j}(z, \bar{a}, \bar{b}), \quad j = 1, \dots, n. \quad (2.4)$$

Considering (2.3) and (2.4) as a holomorphic system of equations with the unknowns \bar{a}, \bar{b} , an application of the implicit function theorem yields holomorphic functions A_1, \dots, A_n, B such that

$$\bar{a}_j = A_j(z, w, w'), \quad \bar{b} = B(z, w, w').$$

The implicit function theorem applies here because the Jacobian of the system coincides with the Levi determinant of M for $(z, w) \in M$ ([1]). Differentiating (2.3) twice and substituting for \bar{a}, \bar{b}

finally yields

$$w_{z_k z_l} = \rho_{z_k z_l}(z, A(z, w, w'), B(z, w, w')) =: \Phi_{kl}(z_1, \dots, z_n, w, w_{z_1}, \dots, w_{z_n}), \quad k, l = 1, \dots, n. \quad (2.5)$$

Now (2.5) is the desired complete system of holomorphic second order PDEs $\mathcal{E} = \mathcal{E}(M)$.

Definition 2.1. We call the PDE system $\mathcal{E} = \mathcal{E}(M)$ *the system of PDEs, associated with M* . We also call the collection $\{\Phi_{ij}\}_{i,j=1}^n$ *the PDE defining function* of a Levi-nondegenerate hypersurface M .

For further developments of the associated differential equations method see, e.g., [27],[26],[24],[25] and references therein.

3. REAL-ANALYTIC HYPERSURFACES IN \mathbb{C}^2 EMBEDDABLE INTO HYPERQUADRICS IN \mathbb{C}^3

In this section we prove Theorem 1 in the more transparent case when $n = 1$ and $N = 2$, so that the source $M \subset \mathbb{C}^2$ and the target quadric $\mathcal{Q} \subset \mathbb{C}^3$. In what follows $(z, w) = (x + iy, u + iv)$ denote the coordinates in \mathbb{C}^2 and (Z_1, Z_2, W) denote that in \mathbb{C}^3 . We start with the observation that, due to the polynomial nature of differential-algebraic operators, *it is sufficient to prove the existence of the desired differential-algebraic operator in a neighborhood of an arbitrary point $p_0 \in M$ when M is given by the same defining equation $v = \varphi(z, \bar{z}, u)$ (at all other points the identity $D(\varphi \equiv 0)$ is satisfied then by analyticity)*. We first write a holomorphic embedding map

$$F = (f_1, f_2, g) : (M, p_0) \mapsto (\mathcal{Q}, F(p_0))$$

for some $p_0 \in M$. Assume F is holomorphic in a small neighborhood U of p_0 in \mathbb{C}^2 . Shifting the base point p_0 , we may assume M to be Levi-nondegenerate at p_0 . We split our arguments into two cases:

Case I: The image of U under F is contained in some affine linear subspace of \mathbb{C}^3 and thus maps M into a hyperquadric in \mathbb{C}^2 (in which case M is biholomorphic to the sphere $S^3 \subset \mathbb{C}^2$);

Case II: The image of U under F is not contained in any affine linear subspace of \mathbb{C}^3 .

The case I of a spherical hypersurface M is considered later separately, so that we assume now to be under the setting of Case II.

By changing the base point and shifting the coordinates, we may assume that $p_0 \in M$ is the origin, and M is Levi-nondegenerate at 0. Let us write the target quadric \mathcal{Q} in the form

$$\text{Im } W = Z_1 \bar{Z}_1 \pm Z_2 \bar{Z}_2.$$

After a change of coordinates in \mathbb{C}^3 preserving the quadric we may assume $F(0) = 0$.

Let us then consider the Segre family $\{S_p\}_{p \in U}$ of M ($U \subset \mathbb{C}^2$ is a neighborhood of the origin). In view of $F(M) \subset \mathcal{Q}$, any Segre variety S_p of a point $p = (a, b) \in U$, considered as a graph $w = w(z) = \rho(z, \bar{a}, \bar{b})$, is contained in the Segre variety of $F(p) = (A, B, C)$. Thus we have,

$$\frac{g - \bar{C}}{2i} = f_1 \bar{A} \pm f_2 \bar{B}|_{w=w(z)}. \quad (3.1)$$

We now differentiate (3.1) three times with respect to z and write the result in terms of the 3-jet

$$(z, w, w', w'', w''')$$

of a Segre variety S_p at a point $(z, w) \in S_p$. Note that each differentiation amounts to applying the vector field

$$\mathcal{L} := \frac{\partial}{\partial z} + w' \frac{\partial}{\partial w}$$

with the rule $\frac{\partial}{\partial z} w^{(j)} = w^{(j+1)}$, $\frac{\partial}{\partial w} w^{(j)} = 0$. Performing the differentiation 3 times, we get:

$$(\mathcal{L}f_1)\bar{A} \pm (\mathcal{L}f_2)\bar{B} - \frac{1}{2i}(\mathcal{L}g) = 0. \quad (3.2)$$

$$(\mathcal{L}^2 f_1)\bar{A} \pm (\mathcal{L}^2 f_2)\bar{B} - \frac{1}{2i}(\mathcal{L}^2 g) = 0. \quad (3.3)$$

$$(\mathcal{L}^3 f_1)\bar{A} \pm (\mathcal{L}^3 f_2)\bar{B} - \frac{1}{2i}(\mathcal{L}^3 g) = 0. \quad (3.4)$$

Considering (3.2)-(3.4) as a linear system for the unknowns $(\bar{A}, \pm\bar{B}, -\frac{1}{2i})$, we conclude that it has a non-zero solution, thus its determinant is 0. That is,

$$\det \begin{pmatrix} \mathcal{L}f_1 & \mathcal{L}f_2 & \mathcal{L}g \\ \mathcal{L}^2 f_1 & \mathcal{L}^2 f_2 & \mathcal{L}^2 g \\ \mathcal{L}^3 f_1 & \mathcal{L}^3 f_2 & \mathcal{L}^3 g \end{pmatrix} \equiv 0. \quad (3.5)$$

Note that

$$\mathcal{L}h = \frac{\partial h}{\partial z} + w' \frac{\partial h}{\partial w};$$

$$\mathcal{L}^2 h = \frac{\partial^2 h}{\partial z^2} + 2w' \frac{\partial^2 h}{\partial z \partial w} + (w')^2 \frac{\partial^2 h}{\partial w^2} + w'' \frac{\partial h}{\partial w};$$

$$\mathcal{L}^3 h = \frac{\partial^3 h}{\partial z^3} + 3w'' \frac{\partial^2 h}{\partial z \partial w} + 3w' \frac{\partial^3 h}{\partial z^2 \partial w} + 3w'' w' \frac{\partial^2 h}{\partial w^2} + 3(w')^2 \frac{\partial^3 h}{\partial z \partial w^2} + (w')^3 \frac{\partial^3 h}{\partial w^3} + w''' \frac{\partial h}{\partial w}.$$

Hence (3.5) reads as

$$w''' P(z, w, w') + R(z, w, w', w'') = 0, \quad (3.6)$$

where P, R are polynomials of the form

$$P = \chi_0 + \chi_1 w' + \chi_2 (w')^2, \quad R = \left(\chi_3 + \chi_4 w' + \chi_5 (w')^2 + \chi_6 (w')^3 + \chi_7 (w')^4 + \right. \\ \left. + \chi_8 (w')^5 + \chi_9 (w')^6 \right) + \left(\chi_{10} + \chi_{11} w' + \chi_{12} (w')^2 + \chi_{13} (w')^3 \right) w'' + \left(\chi_{14} + \chi_{15} w' \right) (w'')^2. \quad (3.7)$$

Here all $\chi_j = \chi_j(z, w)$. We prove the following lemma on χ_j .

Lemma 3.1. *The functions $\chi_j, 0 \leq j \leq 15$, do not all vanish identically.*

Proof. Assume, otherwise, that all $\chi_j, 0 \leq j \leq 15$, are identical zeroes. Then the left hand sides of (3.5) and (3.6) are identically zero when w', w'', w''' are regarded as independent variables:

$$P(z, w, \eta_1) = 0, \quad Q(z, w, \eta_1, \eta_2) = 0 \quad (3.8)$$

for any $\eta_1, \eta_2 \in \mathbb{C}$.

We claim that (3.8) implies the following: for any (fixed) polynomial complex curve

$$\Gamma = \{(z, w) \in \mathbb{C}^2 : w = h(z)\}$$

passing through the origin, the components of the map $F|_\Gamma$ are linearly dependent. Indeed, let us write

$$\Lambda := \frac{\partial}{\partial z} + h'(z) \frac{\partial}{\partial w}$$

as the holomorphic tangent vector of Γ . (Note that applying Λ to a holomorphic function amounts to differentiating it along Γ). By the above observation (3.8), we conclude that

$$\det \begin{pmatrix} \Lambda f_1 & \Lambda f_2 & \Lambda g \\ \Lambda^2 f_1 & \Lambda^2 f_2 & \Lambda^2 g \\ \Lambda^3 f_1 & \Lambda^3 f_2 & \Lambda^3 g \end{pmatrix} = h''' P(z, w, h') + R(z, w, h', h'') = 0 \text{ on } \Gamma. \quad (3.9)$$

By the classical property of Wronskian, we conclude that

$$\lambda_1 \Lambda f_1 + \lambda_2 \Lambda f_2 + \lambda_3 \Lambda g|_\Gamma \equiv 0$$

for some complex numbers $\lambda_i, 1 \leq i \leq 3$, that are not all zero. Furthermore, the assumption $F(0) = 0$ yields $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 g = 0$ on Γ . Thus the components of the map $F|_\Gamma$ are linearly dependent, and this proves the claim.

By the assumption of Case II, there exist three distinct points $p_i = (a_i, b_i), 1 \leq i \leq 3$, near 0 such that

$$\text{Span}_{\mathbb{C}}\{F(p_1), F(p_2), F(p_3)\} = \mathbb{C}^3. \quad (3.10)$$

Perturbing p_i if necessary, we can assume $a_i \neq a_j$ for $i \neq j$ and $a_i \neq 0$ for each i . We then choose a holomorphic polynomial $h_0(z)$ such that $h_0(a_i) = b_i, 1 \leq i \leq 3$, and $h_0(0) = 0$. Hence the origin and p_i are all on the complex curve Γ_0 defined by $w = h_0(z)$. Now the assertion of the above the claim applied for Γ_0 gives a contradiction to (3.10). This establishes the lemma. \square

On the other hand, M is Levi-nondegenerate at 0 and thus its Segre family satisfies a second order ODE

$$w'' = \Phi(z, w, w') \quad (3.11)$$

for a holomorphic near a point $(0, 0, \xi_0)$ function Φ . We now consider the 3-jet space $J^3(\mathbb{C}, \mathbb{C})$ with the coordinates (z, w, ξ, η, ζ) (where ξ, η, ζ correspond to w', w'', w''' respectively) and treat the ODEs (3.6),(3.11) as respectively submanifolds \mathcal{M}, \mathcal{E} in $J^3(\mathbb{C}, \mathbb{C})$. Then \mathcal{M} looks as

$$P(z, w, \xi)\zeta + R(z, w, \xi, \eta) = 0 \quad (3.12)$$

and \mathcal{E} as

$$\eta = \Phi(z, w, \xi), \quad \zeta = \Phi_z + \Phi_w \xi + \Phi_\xi \eta. \quad (3.13)$$

Now the fact that each graph of a solution of (3.11) is contained in that of (3.6) implies

$$\mathcal{E} \subset \mathcal{M},$$

so that

$$\left(\Phi_z + \Phi_w \xi + \Phi_\xi \Phi \right) P(z, w, \xi) + R(z, w, \xi, \Phi) = 0 \quad (3.14)$$

(where $\Phi = \Phi(z, w, \xi)$).

Substituting (3.7) into (3.14), we obtain a scalar *linear* equation for the functions $\chi_0(z, w), \dots, \chi_{15}(z, w)$ with coefficients depending on z, w, ξ . Differentiating this equation 15 time with respect to the variable ξ , we obtain 15 new identities each of which is a similar scalar linear equation for the functions $\chi_0(z, w), \dots, \chi_{15}(z, w)$. In view of the fact that Lemma 3.1, i.e., not all χ_j vanish identically, so that the determinant

$$\mathcal{D}\left(\Phi(z, w, \xi)\right)$$

of the corresponding 16×16 linear system vanishes identically. Note that this determinant is nothing but a *universal differential-algebraic polynomial \mathcal{D} of order 16 applied to the function Φ* . Note that \mathcal{D} is invariant under shifts in z, w . Thus, the embeddability of the hypersurface M into \mathcal{Q} implies

$$\mathcal{D}(\Phi) \equiv 0. \quad (3.15)$$

We finally consider Case I, in which M is spherical. In the latter case, the Segre family of M is locally biholomorphic to the family of straight lines in \mathbb{C}^2 , and hence (see Tresse [35]) $\Phi(z, w, \xi)$ is cubic in the argument ξ (this could be proved by arguments similar to the ones above, but we will not provide here the proof of this classical fact). Now let us write, for the function Φ under consideration, the above determinant $\mathcal{D}(\Phi)$. In view of the fact that Φ is cubic in ξ , the derivatives of the first row of order ≥ 8 vanish identically, so that we conclude that $\mathcal{D}(\Phi) \equiv 0$ in this special case as well.

We shall note that in the latter, equidimensional, case an differential-algebraic operator characterizing the sphericity of a Levi-nondegenerate hypersurface $M \subset \mathbb{C}^2$ was obtained in the work [30] of Merker.

We now need

Proposition 3.2. *The differential-algebraic operator \mathcal{D} is not identical zero.*

Proof. We claim that there exists a function Φ of the form $\Phi = \Phi(\xi)$ such that $\mathcal{D}(\Phi)$ does not vanish identically. Indeed, substituting $\Phi = \Phi(\xi)$ into (3.14) and differentiating 15 times in ξ , we obtain a 16×16 determinant which, in turn, is *the Wronskian of the system of functions in the first row*. This first row has the form

$$\left(\Phi_\xi \Phi, \xi \Phi_\xi \Phi, \xi^2 \Phi_\xi \Phi, 1, \xi, \xi^2, \dots, \xi^6, \Phi, \xi \Phi, \xi^2 \Phi, \xi^3 \Phi, \Phi^2, \xi \Phi^2 \right). \quad (3.16)$$

We then choose an analytic $\Phi(\xi)$ in such a way that the collection of functions in (3.16) is linearly independent (this is possible since every linear dependence between the components of (3.16) implies a nontrivial algebraic or differential equation for Φ). Then the Wronskian $\mathcal{D}(\Phi)$ does not vanish, and this proves the claim and the proposition. \square

Write $w = \rho(z, \bar{z}, \bar{w})$ as the complex defining function of M . We now aim to express the condition $\mathcal{D}(\Phi(z, w, \xi)) = 0$ in the form

$$D^C(\rho(z, a, b)) = 0$$

for some differential-algebraic operator D^C of order 18. Indeed, in view of the Levi-nondegeneracy of M near 0 the map

$$(z, a, b) \mapsto (z, \rho(z, a, b), \rho_z(z, a, b))$$

is a local biholomorphism between $\mathbb{C}^3(z, a, b)$ and $\mathbb{C}^3(z, w, \xi)$. If

$$(z, w, \xi) \mapsto (z, A(z, w, \xi), B(z, w, \xi))$$

is the inverse biholomorphism, then (compare with the associated differential equation procedure described in section 2) we have

$$w = \rho(z, a, b), \quad \xi = \rho_z(z, a, b), \quad \Phi(z, w, \xi) = \rho_{zz}(z, a, b),$$

where $a = A(z, w, \xi)$, $b = B(z, w, \xi)$. Thus Φ is already expressed in terms of ρ . Let us then demonstrate, for example, how we express $\Phi_w(z, w, \xi)$. We have:

$$\Phi_w = \rho_{zza}A_w + \rho_{zzb}B_w. \tag{3.17}$$

We now need to compute A_w, B_w in terms of ρ . For that, we differentiate the identities

$$w = \rho(z, A(z, w, \xi), B(z, w, \xi)), \quad \xi = \rho_z(z, A(z, w, \xi), B(z, w, \xi))$$

in w and get:

$$1 = \rho_a A_w + \rho_b B_w, \quad 0 = \rho_{az} A_w + \rho_{bz} B_w. \tag{3.18}$$

Note that (3.18) is a linear system for A_w, B_w with the determinant at the reference point $(z, w, \xi) = (0, 0, \xi_0)$ being equal to the Levi determinant of M at the origin. Thus this determinant is non-zero and, applying the Cramer rule, we find A_w, B_w as rational functions of the 2-jet of ρ . Substituting into (3.17), we then find Φ_w as a rational function of the 3-jet of ρ .

We then similarly express the entire 16-jet of Φ as a rational (vector-valued) function of the 18-jet of ρ . Thus the condition $\mathcal{D}(\Phi(z, w, \xi)) \equiv 0$ reads as $D^C(\rho(z, a, b)) = 0$ for some order 18 differential-algebraic operator D^C , as required. Note that D^C is invariant under shifts in z, a, b . Also note that the differential-algebraic operator D^C is not identical zero. Indeed, by Proposition 3.2, there is a $\Phi(z, w, w')$ with $\mathcal{D}(\Phi) \neq 0$, and for such $\Phi(z, w, \xi)$ we find $\rho(z, a, b)$ with $D^C(\rho) = \mathcal{D}(\Phi) \neq 0$ by solving the ODE $w'' = \Phi(z, w, w')$ with the initial data $w(0) = a, w'(0) = b$.

Now it is not difficult to complete the proof of Theorem 1 in the case of CR-dimension 1.

Proof of Theorem 1 for $n = 1, N = 2$.

Recall that the complex defining function $\rho(z, a, b)$ and the real defining function $\varphi(z, a, u)$ are connected via the identities:

$$\rho(z, a, b) = u + i\varphi(z, a, u), \quad b = u - i\varphi(z, a, u).$$

Then, for example, the derivative ρ_b is expressed via the 1-jet of φ as follows: we have

$$\rho_b = u_b + iu_b\varphi_u, \quad 1 = u_b - iu_b\varphi_u,$$

so that

$$\rho_b = (1 + i\varphi_u)/(1 - i\varphi_u).$$

We similarly expressed the entire 18-jet of ρ as a rational (vector-valued) function of the 18-jet of φ . Thus the condition $D^C(\rho(z, w, \xi)) \equiv 0$ reads as $D(\varphi(z, a, b)) = 0$ for some order 18 differential-algebraic operator D .

It remains to show that the operator D is non-trivial (on the space of real-analytic defining functions φ of real hypersurfaces). For that we note that D is identical zero if and only if it is identical zero on the subspace of φ defining a real hypersurface (since the latter subspace is totally real). However, $D(\varphi)$ is not identical zero since $D^C(\rho)$ is not identical zero, as was shown above. This proves the theorem for $n = 1, N = 2$. □

Proof of Theorem 2 for $n = 1, N = 2$. Let us denote by V_k the space of polynomials $\varphi(z, a, u)$ of degree $\leq k$ for some $k \geq 18$. We claim that there exists $\varphi \in V_k$ such that $D(\varphi)$ does not vanish identically. Indeed, the identity $D(\varphi) = 0$ defines a proper algebraic variety A in the jet bundle $J^{18}(\mathbb{C}^3, \mathbb{C})$ (the properness follows from the non-triviality of D). Picking a point $q \in J^{18}(\mathbb{C}^3, \mathbb{C}) \setminus A$ we choose the unique polynomial $\psi \in V_k$ of degree 18 with the 18-jet corresponding to q , and get $D(\psi) \neq 0$, as required. Thus D is generically non-vanishing on V_k .

If we now consider the set W_k of $\varphi(z, a, u)$ arising from an algebraic equation $P(z, a, u, v) = 0$ for a polynomial P of degree $\leq k$, then W_k has a structure of algebraic manifold. Hence either D vanishes identically on W_k , or is (Zariski) generically non-vanishing. By the above argument, we conclude that D is (Zariski) generically non-vanishing on W_k , and this implies the claim of the theorem for $n = 1, N = 2$. □

4. THE HIGH DIMENSIONAL CASE

In this section Theorem 1 and Theorem 2 will be established in the general case.

For a fixed $n \geq 1$, we set \mathcal{M}_m , $m \geq n$, to be the set of all Levi-nondegenerate hypersurfaces in \mathbb{C}^{n+1} that can be locally transversally holomorphically embedded into a hyperquadric $\mathcal{Q}^{2m+1} \subset \mathbb{C}^{m+1}$. We also write $\tilde{\mathcal{M}}_m \subset \mathcal{M}_m$ to be the collection of Levi-nondegenerate hypersurfaces M in \mathbb{C}^{n+1} satisfying the following property (*) (for a fixed m):

Property (*) *There exist a point $p \in M$ and a holomorphic map F from a small neighborhood U of p to \mathbb{C}^{m+1} such that M is locally transversally holomorphically embedded into $\mathcal{Q}^{2m+1} \subset \mathbb{C}^{m+1}$ by F . Moreover, the image of U under F is not contained in any affine linear subspace of \mathbb{C}^{m+1} .*

Note that when $m = n$, the assumption in Property (*) that the image of U is not contained in any affine linear subspace of \mathbb{C}^{m+1} can be dropped, as it follows already from the transversality assumption. We obviously have

$$\bigcup_{m=n}^N \tilde{\mathcal{M}}_m = \mathcal{M}_N. \quad (4.1)$$

We prove in this section the following two theorems implying Theorem 1 and Theorem 2 respectively.

Theorem 3. *For any integers $m \geq n \geq 1$, there exists an universal non-zero shift-invariant differential-algebraic operator $D = D(n, m)$ such that the following holds. If a germ of real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ with a defining equation*

$$v = \varphi(z, \bar{z}, u)$$

is contained in $\tilde{\mathcal{M}}_m$, i.e., satisfies property () for m , then*

$$D(\varphi) \equiv 0.$$

Theorem 4. *For any pair of integers $m \geq n \geq 1$, there exists a positive integer $\nu = \nu(n, m)$ such that a Zariski generic real-algebraic hypersurface $M \subset \mathbb{C}^{n+1}$ of any degree $\geq \nu$ is not contained in $\tilde{\mathcal{M}}_m$.*

In what follows $(z, w) = (z_1, \dots, z_n, w)$ denote the coordinates in \mathbb{C}^{n+1} and (Z_1, \dots, Z_m, W) denote that in \mathbb{C}^{m+1} . Write the holomorphic embedding map

$$F = (f_1, \dots, f_m, g) : (M, p_0) \rightarrow (\mathcal{Q}^{2m+1}, F(p_0)),$$

for some Levi-nondegenerate point p_0 . As in Section 3, by shifting the base point of the coordinates, we can assume $p_0 = 0$.

Let us write the target hyperquadric $\mathcal{Q}^{2m+1} = \mathcal{Q}_l^{2m+1}$ in the form

$$\operatorname{Im}W = -Z_1\bar{Z}_1 - \dots - Z_l\bar{Z}_l + Z_{l+1}\bar{Z}_{l+1} + \dots + Z_m\bar{Z}_m,$$

where l is the signature of \mathcal{Q}^{2m+1} .

Let us now consider the Segre family $\{S_p\}$ of M . In view of $F(M) \subset \mathcal{Q}^{2m+1}$, any Segre variety S_p of a point $p = (a, b) = (a_1, \dots, a_n, b) \in U$, considered as a graph $w = w(z) = \rho(z, \bar{a}, \bar{b})$ is contained in the Segre variety of $F(p) = (A_1, \dots, A_m, C)$. Thus we have:

$$\frac{g - \bar{C}}{2i} = -f_1\bar{A}_1 - \dots - f_l\bar{A}_l + f_{l+1}\bar{A}_{l+1} + \dots + f_m\bar{A}_m|_{w=w(z)}. \quad (4.2)$$

As in Section 3, we differentiate several times with respect to z and write the result in terms of the μ -jet

$$(z, w, w^{(\alpha)})_{1 \leq |\alpha| \leq \mu}$$

of a Segre variety S_p at a point $(z_1, \dots, z_n, w) \in S_p$ for some μ . Here we use the notation

$$w^{(\alpha)} = \frac{\partial^{|\alpha|} w}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}$$

for any multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$. For first order derivatives we use the notation $w'_j = \frac{\partial w}{\partial z_j}$.

For the following, we fix

$$k := m - n + 1.$$

Write the basis of holomorphic tangent vectors along S_p as

$$\mathcal{L}_j = \frac{\partial}{\partial z_j} + w'_j \frac{\partial}{\partial w}, 1 \leq j \leq n.$$

Next, for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we write $\mathcal{L}^\alpha = \mathcal{L}_1^{\alpha_1} \dots \mathcal{L}_n^{\alpha_n}$. We apply $\mathcal{L}^{\alpha^1}, \dots, \mathcal{L}^{\alpha^m}, \mathcal{L}^{\alpha^{m+1}}$ to equation (4.2) for some multi-indices $\alpha^1, \dots, \alpha^{m+1}$ (precise form of which will be determined later) with each

$$|\alpha^i| \leq m - n + 2 = k + 1.$$

As the result of the differentiations, we obtain:

$$-\mathcal{L}^{\alpha^j} \bar{A}_1 - \dots - \mathcal{L}^{\alpha^j} f_l \bar{A}_l + \mathcal{L}^{\alpha^j} f_{l+1} \bar{A}_{l+1} + \dots + \mathcal{L}^{\alpha^j} f_m \bar{A}_m - \frac{1}{2i} (\mathcal{L}^{\alpha^j} g) = 0, \quad 1 \leq j \leq m+1. \quad (4.3)$$

Equations (4.3) form a linear system for $(m+1)$ unknowns $(-\bar{A}_1, \dots, -\bar{A}_l, \bar{A}_{l+1}, \dots, \bar{A}_m, -\frac{1}{2i})$. Note that it has a non-zero solution, thus its determinant is 0. That is,

$$\det \begin{pmatrix} \mathcal{L}^{\alpha^1} f_1 & \dots & \mathcal{L}^{\alpha^1} f_m & \mathcal{L}^{\alpha^1} g \\ \dots & \dots & \dots & \dots \\ \mathcal{L}^{\alpha^m} f_1 & \dots & \mathcal{L}^{\alpha^m} f_m & \mathcal{L}^{\alpha^m} g \\ \mathcal{L}^{\alpha^{m+1}} f_1 & \dots & \mathcal{L}^{\alpha^{m+1}} f_m & \mathcal{L}^{\alpha^{m+1}} g \end{pmatrix} \equiv 0 \text{ on } S_p. \quad (4.4)$$

Note that for a function h we have

$$\mathcal{L}_j h = \frac{\partial h}{\partial z_j} + w'_j \frac{\partial h}{\partial w};$$

$$\mathcal{L}_i \mathcal{L}_j h = \frac{\partial^2 h}{\partial z_i \partial z_j} + w'_i \frac{\partial^2 h}{\partial z_j \partial w} + w'_j \frac{\partial^2 h}{\partial z_i \partial w} + w'_i w'_j \frac{\partial^2 h}{\partial w^2} + w''_{ij} \frac{\partial h}{\partial w}.$$

In general, for a multiindex α_j , $\mathcal{L}^{\alpha^j} h$ is a polynomial in $w^{(\beta)}$, $1 \leq |\beta| \leq |\alpha^j|$, of degree $|\alpha^j|$ with coefficients in the jet space of h ; that is why the left hand side of (4.4) has the form $H(z, w, (w^{(\beta)})_{1 \leq |\beta| \leq k+1})$ and is polynomial in $w^{(\beta)}$, $1 \leq |\beta| \leq k+1$, with coefficients in the $(k+1)$ -jet of F . Hence (4.4) reads as

$$H(z, w, (w^{(\beta)})_{1 \leq |\beta| \leq k+1}) =: \eta_0(z, w) h_0((w^{(\beta)})_{1 \leq |\beta| \leq k+1}) + \dots + \eta_s(z, w) h_s((w^{(\beta)})_{1 \leq |\beta| \leq k+1}) = 0. \quad (4.5)$$

Here $\{h_1, \dots, h_s\}$ is the collection of all distinct monomials in $(w^{(\beta)})_{1 \leq |\beta| \leq k+1}$ of degree at most d , where $d = |\alpha^1| + \dots + |\alpha^{m+1}|$. The coefficients $\eta_j(z, w)$ are certain functions, polynomially depending on $j^{k+1}F$ (the latter dependence is fixed by the choice of $\alpha^{n+1}, \dots, \alpha^{m+1}$).

We prove the following lemma on η_j 's. Write for each $1 \leq i \leq n$, we write

$$\epsilon^i = (0, \dots, 0, 1, 0, \dots, 0),$$

where the component "1" is at the i^{th} position, so that $\mathcal{L}^{\epsilon^i} = \mathcal{L}_i, 1 \leq i \leq n$.

Lemma 4.1. *We can choose multi-indices $\alpha^i, 1 \leq i \leq m+1$ in such a way that not all $\eta_j, 1 \leq j \leq s$, in (4.5) are identical zeroes. Moreover, we can achieve $\alpha^i = \epsilon^i$ for $1 \leq i \leq n$, and $|\alpha^i| \leq i - (n-1)$ for $n+1 \leq i \leq m+1$.*

Proof. Suppose, otherwise, that for $\alpha^i = \epsilon^i, 1 \leq i \leq n$, and any choices of multi-indices $\alpha^i, n+1 \leq i \leq m+1$ with each $|\alpha^i| \leq i - (n-1)$, we always get all $\eta_j, 1 \leq j \leq s$, identically zero. This means H in (4.5) is identical zero when $w^{(\beta)}$ are regarded as independent variables, for any such choice of α^i 's.

Claim: Let Γ be any complex manifold in \mathbb{C}^{n+1} defined by $w = h(z)$ passing through the origin, where $h(z)$ is a holomorphic polynomial in z with $h(0) = 0, \frac{\partial h}{\partial z_j}(0) = 0$ for all $1 \leq j \leq n$. Then the components of F are linearly dependent over \mathbb{C} on Γ .

Proof of Claim: Write $\Lambda_j = \frac{\partial}{\partial z_j} + \frac{\partial h(z)}{\partial z_j} \frac{\partial}{\partial w}, 1 \leq j \leq n$. Note that $\Lambda_j|_0 = \frac{\partial}{\partial z_j}$. Then we have, since F is an embedding,

$$\dim_{\mathbb{C}}(\text{Span}_{\mathbb{C}}\{\Lambda_1 F(q), \dots, \Lambda_n F(q)\}) = n \quad (4.6)$$

for any point q near 0 on Γ .

However, by the hypotheses that H (which, we recall, equals to the determinant (4.4)) is identically zero, for any choice of multiindices $\alpha^i, n+1 \leq i \leq m+1$ with $|\alpha^i| \leq i - (n-1) \forall i$, we have on Γ :

$$\dim_{\mathbb{C}}(\text{Span}_{\mathbb{C}}\{\Lambda_1 F(q), \dots, \Lambda_n F(q), \Lambda^{\alpha^{n+1}} F(q), \dots, \Lambda^{\alpha^{m+1}} F(q)\}) < m+1. \quad (4.7)$$

We then have the following proposition, which can be regarded as a generalization of Wolsson's result [36].

Proposition 4.2. *Under the assumptions of (4.6) and (4.7) for any choices of multiindices $\alpha^i, n+1 \leq i \leq m+1$ with each $|\alpha^i| \leq i - (n-1)$, we conclude that there exists $\lambda_1, \dots, \lambda_{m+1}$ that are not all zero such that*

$$\lambda_1 \Lambda_j f_1 + \dots + \lambda_m \Lambda_j f_m + \lambda_{m+1} \Lambda_j g = 0$$

on Γ for all $1 \leq j \leq n$ at once.

Proof of Proposition 4.2. When $n = 1$, the result follows from the result of Wolsson [36]. In the general dimensional case, Proposition 4.2 essentially follows from the framework in the paper of Berhanu and the second author [4]. To make the paper more self-contained, we include a proof in Appendix II. \square

We return to the proof of the Claim. By Proposition 4.2, the expression

$$\lambda_1 f_1 + \cdots + \lambda_m f_m + \lambda_{m+1} g$$

is a constant on Γ (since all its partial derivatives vanish on Γ). As we have $F(0) = 0$, we finally conclude that the components of F are linearly dependent on Γ . This proves the claim. ■

End of the proof of Lemma 4.1. Now, since M, F are as in the definition of $\tilde{\mathcal{M}}_m$, the image of U under F is not contained in any affine linear subspace of \mathbb{C}^{m+1} . There exist $m+1$ points $p_j = (a_j^1, \dots, a_j^n, b_j), 1 \leq j \leq m+1$, near 0 such that

$$\text{Span}_{\mathbb{C}}\{F(p_1), \dots, F(p_{m+1})\} = \mathbb{C}^{m+1}. \quad (4.8)$$

Perturbing p_j 's if necessary, we can assume that $a_i^1 \neq a_j^1$ if $i \neq j$ and $a_j^1 \neq 0$ for all $1 \leq j \leq m+1$. Let $h_1(z_1)$ be the holomorphic polynomial in z_1 such that $h_1(a_j^1) = b_j, 1 \leq j \leq m+1, h_1(0) = 0$. Let $h_2(z_1)$ be the holomorphic polynomial in z_1 such that $h_2(a_j^1) = 1, 1 \leq j \leq m+1, h_2(0) = 0$. Set $h_0(z) = h_1(z_1)h_2(z_1)$. Let Γ_0 be the complex curve defined by $w = h_0(z)$. Then Γ_0 satisfies the assumptions in the claim. Moreover, all $p_j, 1 \leq j \leq m+1$, are on Γ_0 . We then apply the result in the claim to get a contradiction with (4.8). Thus Lemma 4.1 is established. □

In what follows, we assume $\alpha^1, \dots, \alpha^{m+1}$ to be chosen as desired in Lemma 4.1.

On the other hand, M is Levi-nondegenerate at 0 and thus its Segre family satisfies a completely integrable system of second order PDEs:

$$w''_{ij} = \Phi_{ij}(z, w, w'_1, \dots, w'_n), \quad 1 \leq i, j \leq n, \quad (4.9)$$

for holomorphic near a point $(0, \xi_1^0, \dots, \xi_n^0)$ functions

$$\{\Phi_{ij}\}, \quad i, j = 1, \dots, n.$$

We now regard both the $(k+1)$ -jet prolongation of (4.5) and the PDE system (4.9) as submanifolds \mathcal{M}, \mathcal{E} respectively in the $(k+1)$ -jet space $J^{k+1}(\mathbb{C}^n, \mathbb{C})$ with the coordinates $(z, w, \xi_\alpha)_{1 \leq |\alpha| \leq k+1}$, where α is a multiindex running through all $1 \leq |\alpha| \leq k+1$ (here ξ_α corresponds to $w^{(\alpha)}$; in particular, ξ_l corresponds to $\frac{\partial w}{\partial z_l}$). Then \mathcal{M} looks as

$$H(z, w, (\xi_\alpha)_{1 \leq |\alpha| \leq k+1}) = \eta_0(z, w)h_0((\xi_\alpha)_{1 \leq |\alpha| \leq k+1}) + \cdots + \eta_s(z, w)h_s((\xi_\alpha)_{1 \leq |\alpha| \leq k+1}) = 0 \quad (4.10)$$

and \mathcal{E} as

$$\begin{aligned}
\xi_{ij} &= \Phi_{ij}(z, w, \xi); \\
\xi_{ijk} &= (\Phi_{ij})_{z_k} + (\Phi_{ij})_w \xi_k + \sum_{l=1}^n (\Phi_{ij})_{\xi_l} \Phi_{lk}; \\
&\dots\dots \\
\xi_\alpha &= Q_\alpha\left(\xi, (\Phi_{ij}^{(\beta)})_{|\beta|\leq|\alpha|-2}\right); \\
&\dots\dots \\
\xi_{\alpha^{m+1}} &= Q_{\alpha^{m+1}}\left(\xi, (\Phi_{ij}^{(\beta)})_{|\beta|\leq k-1}\right) =: \tilde{Q}\left(\xi, (\Phi_{ij}^{(\beta)})_{|\beta|\leq k-1}\right).
\end{aligned} \tag{4.11}$$

Here $\xi = (\xi_1, \dots, \xi_n)$, all Q_α 's are certain *universal* polynomials in their arguments, and we use the notation

$$\xi_{ij} := \xi_{(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)},$$

where $(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$ is a multiindex with the two 1's at the i^{th} and j^{th} positions, and similarly for ξ_{ijk} .

We substitute all ξ_α 's in h_j by Q_α to obtain the polynomials h_j purely in terms of ξ and $\{\Phi_{ij}\}_{1 \leq i, j \leq n}$ and their derivatives with respect to z, w, ξ . The new polynomials are denoted by

$$\tilde{h}_j(\xi, (\Phi_{ij}^{(\beta)})_{|\beta|\leq k-1}). \tag{4.12}$$

Arguing now as in Section 3 and considering the submanifolds \mathcal{E}, \mathcal{M} of an appropriate jet space corresponding to the PDE systems (4.11), (4.10) respectively, we write up the fact that $\mathcal{E} \subset \mathcal{M}$ and obtain:

$$\tilde{H}(z, w, \xi, (\Phi_{ij}^{(\beta)})_{|\beta|\leq k-1}) := \eta_0(z, w) \tilde{h}_0(\xi, (\Phi_{ij}^{(\beta)})_{|\beta|\leq k-1}) + \dots + \eta_s(z, w) \tilde{h}_s(\xi, (\Phi_{ij}^{(\beta)})_{|\beta|\leq k-1}) = 0. \tag{4.13}$$

The condition (4.13) gives us a scalar linear equation for $\eta_0(z, w), \dots, \eta_s(z, w)$ with coefficients depending on z, w, ξ . We then choose a collection of pairwise distinct multiindices

$$\gamma^1, \dots, \gamma^s \in (\mathbb{Z}_{\geq 0})^n \quad \text{with} \quad 1 \leq |\gamma^j| \leq j,$$

and perform s differentiations of the above scalar linear equation by means of the differential operators

$$\frac{\partial^{|\gamma^1|}}{\partial \xi^{\gamma^1}}, \dots, \frac{\partial^{|\gamma^s|}}{\partial \xi^{\gamma^s}}.$$

Thus we obtain s new identities each of which is a scalar linear equation for the functions χ_0, \dots, χ_s , and this gives us an $(s+1) \times (s+1)$ linear system. Recall again χ_j 's do not all vanish identically, so that the determinant of this system, which we write as

$$\mathcal{D}(\alpha^1, \dots, \alpha^{m+1} | \gamma^1, \dots, \gamma^s) (\{\Phi_{ij}(z, w, \xi)\}_{1 \leq i, j \leq n}), \tag{4.14}$$

vanishes identically, where $\mathcal{D}(\alpha^1, \dots, \alpha^{m+1} | \gamma^1, \dots, \gamma^s)$ is an differential-algebraic operator. We shall now prove the following

Proposition 4.3. *There exist multiindices $\{\gamma^1, \dots, \gamma^s\}$ with $1 \leq |\gamma^j| \leq j, 1 \leq j \leq s$ such that $\mathcal{D}(\alpha^1, \dots, \alpha^{m+1} | \gamma^1, \dots, \gamma^s)$ is not identical zero.*

Proof. Similarly as in the case $n = 1$, we consider systems of the kind (4.9) with the right hand side depending on the derivatives w'_1, \dots, w'_n only, so that the right hand side in (4.11) depends on ξ only (and does not depend on z, w). Then we consider the first row in the $(s+1) \times (s+1)$ determinant (4.14):

$$(\tilde{h}_0(\xi, (\Phi_{ij}^{(\beta)})_{|\beta| \leq k-1}), \dots, \tilde{h}_s(\xi, (\Phi_{ij}^{(\beta)})_{|\beta| \leq k-1})). \quad (4.15)$$

We claim that we can choose analytic functions $\{\Phi_{ij}\}_{1 \leq i, j \leq n}$ in such a way that the components of (4.15) are linearly independent. To prove the claim, we choose a holomorphic function

$$w = w^*(z) : (\mathbb{C}^n, 0) \mapsto (\mathbb{C}, 0)$$

with the following property: $w^*(z)$ does not satisfy any differential-algebraic equation (the existence of such functions is well known since the classical work of Ostrowski [31]). By moving to a generic point p near 0 and applying a linear change of coordinates to make $p = 0$, we can assume $(w^*_{z_i z_j}(0))_{1 \leq i, j \leq n}$ is nondegenerate. Then we can express each z_l as a function of $\{w^*_{z_j}\}_{j=1}^n$ near 0. Next, we choose a complete system of the kind (4.9) (with the defining function $\{\Phi_{ij}^*\}$ depending on ξ only, as discussed above), having $w^*(z)$ as a solution: one can construct Φ_{ij}^* by expressing, for example, each z_l as a function of $\{w^*_{z_j}\}_{j=1}^n$ and substituting the result into $w^*_{z_i z_j}(z)$. We then observe that, since $w^*(z)$ is a solution of the system (4.9) with the defining function $\{\Phi_{ij}^*\}$, then evaluating a monomial $h_j((w^{(\beta)})_{1 \leq |\beta| \leq k+1})$ at the $(k+1)$ jet of the function $w = w^*(z)$ amounts (by the definition of $h_j(\xi, (\Phi_{ij}^{(\beta)})_{|\beta| \leq k-1})$) to substituting the $(k+1)$ jet of $w = w^*(z)$ into $h_j(\xi, (\Phi_{ij}^{(\beta)})_{|\beta| \leq k-1})$. Now, assume that for the above choice of the defining function in (4.9) there is a non-trivial linear dependence between the components of the first row of the determinant (4.14). Then we conclude that the same non-trivial linear dependence holds for the monomials $h_j((w^{(\beta)})_{1 \leq |\beta| \leq k+1})$ evaluated at the $(k+1)$ jet of the function $w = w^*(z)$. Since all the latter monomials are *distinct*, we obtain a non-trivial differential-algebraic equation for the function $w = w^*(z)$, which gives a contradiction and proves the claim.

Now, using the above choice of the defining function in (4.9), we make use of a result of Wolsion ([36]) which states that there exists a non-vanishing identically generalized Wronskians of the components of the first row, and this yields the existence of the desired multiindices $\{\gamma^1, \dots, \gamma^s\}$. \square

We have now an important

Remark 4.4. In fact, the proof of Proposition 4.3 implies a stronger fact, which is the non-triviality of the restriction of the operator \mathcal{D} constructed in Proposition 4.3 onto the subset I of all possible analytic right hand sides $(\{\Phi_{ij}\}_{1 \leq i, j \leq n})$ corresponding to completely integrable systems (4.9).

Further, we emphasize the following.

Remark 4.5. The operator $\mathcal{D}(\alpha^1, \dots, \alpha^{m+1} | \gamma^1, \dots, \gamma^s)$ constructed in Proposition 4.3 is universal, in the sense that it depends on $\{\alpha^1, \dots, \alpha^{m+1}\}$ and $\{\gamma^1, \dots, \gamma^s\}$ only. In turn, $\{\gamma^1, \dots, \gamma^s\}$ are determined

by $\{\alpha^1, \dots, \alpha^{m+1}\}$, that is why we write in short

$$\mathcal{D}(\alpha^1, \dots, \alpha^{m+1}) = \mathcal{D}(\alpha^1, \dots, \alpha^{m+1} | \gamma^1, \dots, \gamma^s)$$

in what follows. We also remark that for each $\{\alpha^1, \dots, \alpha^{m+1}\}$, the order of derivatives of $\{\Phi_{ij}\}_{1 \leq i, j \leq n}$ that appears in $\mathcal{D}(\alpha^1, \dots, \alpha^{m+1})$ is at most $s + k - 1 \leq s + m - n$. Indeed, this can be easily seen from (4.12) and the fact that $|\gamma^j| \leq s$ for all $1 \leq j \leq s$ in Proposition 4.3. Thus, the order d of the differential-algebraic operator $\mathcal{D}(\alpha^1, \dots, \alpha^{m+1})$ satisfies

$$d \leq s + m - n. \quad (4.16)$$

Now to obtain a differential-algebraic operator annihilating the right hand side of any nondegenerately embeddable hypersurface, we argue as follows. For a holomorphic nondegenerate embedding map $F : M \rightarrow \mathcal{Q}^{2m+1}$ defined near 0 we consider all possible choices of $\{\alpha^1, \dots, \alpha^{m+1}\}$, where they are as required in Lemma 4.1. In particular there exist finitely many such choices of $\{\alpha^1, \dots, \alpha^{m+1}\}$. Then we obtain a collection of finitely many operators

$$\{\mathcal{D}_1, \dots, \mathcal{D}_\nu\}$$

such that for any $M \subset \tilde{\mathcal{M}}_m$, there exists $l \leq \nu$ with $\mathcal{D}_l(\{\Phi_{ij}\}_{1 \leq i, j \leq n}) = 0$. Now the product operator

$$\mathcal{D}(\{\Phi_{ij}\}_{1 \leq i, j \leq n}) := \mathcal{D}_1(\{\Phi_{ij}\}_{1 \leq i, j \leq n}) \cdots \mathcal{D}_\nu(\{\Phi_{ij}\}_{1 \leq i, j \leq n}). \quad (4.17)$$

satisfies $\mathcal{D}(\{\Phi_{ij}\}_{1 \leq i, j \leq n}) \equiv 0$ if $\{\Phi_{ij}\}$ is associated to some $M \subset \tilde{\mathcal{M}}_m$.

The next step in proving Theorem 3 is to transfer to the complex defining function $\rho(z, a, b)$, which we do similarly to the case $n = 1$. Our goal is to express $\mathcal{D}(\{\Phi_{ij}\}_{1 \leq i, j \leq n})$ as a rational function of the $(d + 2)$ -jet of ρ (where d is the order of \mathcal{D}).

In view of the Levi-nondegeneracy of M near 0, the map

$$(z, a, b) \mapsto (z, \rho(z, a, b), \rho_z(z, a, b))$$

is a local biholomorphism between $\mathbb{C}^{2n+1}(z, a, b)$ and $\mathbb{C}^{2n+1}(z, w, \xi)$. Here we write $z = (z_1, \dots, z_n)$, $a = (a_1, \dots, a_n)$. If

$$(z, w, \xi) \mapsto (z, A(z, w, \xi), B(z, w, \xi))$$

is the inverse biholomorphism, where we write $A(z, w, \xi) = (A_1(z, w, \xi), \dots, A_n(z, w, \xi))$, then we have

$$w = \rho(z, a, b), \xi_i = \rho_{z_i}(z, a, b), \Phi_{ij}(z, w, \xi) = \rho_{z_i z_j}(z, a, b), 1 \leq i, j \leq n, \quad (4.18)$$

where $a = A(z, w, \xi)$, $b = B(z, w, \xi)$. Thus Φ_{ij} is already expressed in terms of the derivatives of ρ . Then let us demonstrate the way to express, for instance, $(\Phi_{ij})_{z_k}$, $1 \leq k \leq n$. First,

$$(\Phi_{ij})_{z_k} = \rho_{z_i z_j z_k} + \sum_{l=1}^n \rho_{z_i z_j a_l} (A_l)_{z_k} + \rho_{z_i z_j b} B_{z_k}. \quad (4.19)$$

For each fixed k , we now need to compute $(A_l)_{z_k}, B_{z_k}, 1 \leq l \leq n$, in terms of ρ and its derivatives. For that we differentiate the first two equations in (4.18) with respect to z_k and get,

$$\begin{aligned} 0 &= \rho_{z_k} + \sum_{l=1}^n \rho_{a_l} (A_l)_{z_k} + \rho_b B_{z_k} \\ 0 &= \rho_{z_i z_k} + \sum_{l=1}^n \rho_{z_i a_l} (A_l)_{z_k} + \rho_{z_i b} B_{z_k}, 1 \leq i \leq n \end{aligned} \tag{4.20}$$

Note that (4.20) is a linear system for $(A_l)_{z_k}, B_{z_k}, 1 \leq l \leq n$ whose determinant at the reference point $(0, 0, \xi^0)$ being equal to the Levi determinant of M at the origin. By the Levi-nondegeneracy and applying Cramer's rule, we can solve $(A_l)_{z_k}, B_{z_k}, 1 \leq l \leq n$ as rational functions of 2-jet of ρ . Substituting into (4.19), we get $(\Phi_{ij})_{z_k}$ as rational functions of the 3-jet of ρ . We similarly express the entire d -jet of Φ_{ij} as a rational function of the $(d+2)$ -jet of ρ . In this manner we obtain an algebraic differential operators D^C , such that the condition $\mathcal{D}(\{\Phi_{ij}\}_{1 \leq i, j \leq n}) \equiv 0$ reads as $D^C(\rho(z, a, b)) \equiv 0$.

Furthermore, we can apply an argument similar to that in Section 3 to find an differential-algebraic operator $D = D(n, m)$ of order $d+2$ such that $D^C(\rho) \equiv 0$ reads as $D(\varphi) \equiv 0$, where $\varphi(z, a, u)$ is the real defining function of a hypersurface M .

Finally, it remains to show that D is non-trivial on the space of real-analytic defining functions $\varphi(z, a, u)$ of real hypersurfaces. Since the latter subspace is totally real, it is enough to show the non-triviality of D on the space of all possible analytic $\varphi(z, a, u)$, which is equivalent to the non-triviality of $D^C(\rho)$ on the space of all possible analytic $\rho(z, a, b)$. The desired non-triviality of $D^C(\rho)$ amounts to the non-triviality of \mathcal{D} on the subspace I of all possible analytic right hand sides $(\{\Phi_{ij}\}_{1 \leq i, j \leq n})$ corresponding to completely integrable systems (4.9) and hence follows from Remark 4.4. This completes the proof of Theorem 3. \square

It is not difficult now to verify the proof of Theorem 1.

Proof of Theorem 1. Recall that we have the decomposition (4.1). Thus the desired differential-algebraic operator is given as the product

$$D(n, n)(\varphi) \cdots D(n, N)(\varphi). \tag{4.21}$$

As follows from the construction, (4.21) annihilates real defining functions of all hypersurfaces from \mathcal{M}_N and is shift-invariant. This completes the proof of Theorem 1. \square

Arguing then identically to the proof of Theorem 2 in the case $n = 1$, we obtain the proof of Theorem 4 with $\nu(n, m)$ being the order of the differential operator in Theorem 3. We shall note that, as follows from (4.17), the differential operator in Theorem 1 is obtained by multiplying several lower order operators. Using this observation, we can improve the bound for $\nu(n, m)$ to being equal to

$$\nu(n, m) = 2 + \max \left\{ \text{ord } \mathcal{D}(\alpha^1, \dots, \alpha^m) \right\}, \tag{4.22}$$

where $\alpha^1, \dots, \alpha^m \in (\mathbb{Z}_{\geq 0})^n$ are all distinct and satisfy the requirement in Lemma 4.1. More precisely, we require for $1 \leq i \leq n, \alpha^i = \epsilon^i$, and for $n+1 \leq i \leq m+1, |\alpha^i| \leq i - (n-1)$.

It is also immediate to see from (4.21) and (4.1) that Theorem 4 implies Theorem 2 with $\mu(n, N)$ being equal to

$$\mu(n, N) = \max \left\{ \nu(n, m) \right\}_{m \in [n, N]}. \quad (4.23)$$

Explicit bounds for $\nu(n, m), \mu(n, m)$ are given in Appendix I below.

In the end of this section, we would like to emphasize that it looks particularly interesting to study, by using the method of the present paper, the problem of characterization of the class of real hypersurfaces that can be holomorphically embedded into the particular hyperquadric \mathbb{S}^{2N+1} (namely, the sphere). On the latter problem, see [21],[15], [20], [22]. In particular, [22] shows that the examples of compact real algebraic strictly pseudoconvex hypersurfaces constructed in [20] are not holomorphically embeddable into any spheres. We conjecture here that the property of the embeddability into a sphere is characterized by differential-algebraic inequalities, supplementing the differential-algebraic equations introduced in the paper. (Such inequalities should arise from the Cramer's rule applied to a linear system of the kind (4.3)).

5. APPENDIX I

In this section we obtain explicit upper bounds for $\nu(n, m), \mu(n, m)$ in Theorem 4 and Theorem 2 respectively.

In view of (4.22),(4.23) and (4.16), obtaining upper bounds for $\nu(n, m), \mu(n, m)$ amounts to estimating, for each $\{\alpha^1, \dots, \alpha^m\}$ as above, the total number s of terms in (4.13). We first introduce,

Definition 5.1. Let $\lambda w^{(\beta^1)} \dots w^{(\beta^l)}, l \geq 1$, be a monomial in the derivatives of w . We define the *weighted degree* of this monomial to be $|\beta^1| + \dots + |\beta^l|$. If W is a polynomial which is a sum of monomials of such form, then the weighted degree of W is defined to be the highest weighted degree of these monomials.

Lemma 5.2. Let $\mathcal{L}_j, 1 \leq j \leq n$ be as in Section 4. Let h be one of the functions f_1, \dots, f_m, g . For any multiindex α , $\mathcal{L}^\alpha h$ is a polynomial in $\{w^{(\beta)}\}_{|\beta| \leq |\alpha|}$ of weighted degree $|\alpha|$.

Proof. Note that

$$\begin{aligned} \mathcal{L}_j h &= \frac{\partial h}{\partial z_j} + w'_j \frac{\partial h}{\partial w}; \\ \mathcal{L}_i \mathcal{L}_j h &= \frac{\partial^2 h}{\partial z_i \partial z_j} + w'_i \frac{\partial^2 h}{\partial z_j \partial w} + w'_j \frac{\partial^2 h}{\partial z_i \partial w} + w'_i w'_j \frac{\partial^2 h}{\partial w^2} + w''_{ij} \frac{\partial h}{\partial w}. \end{aligned}$$

Hence the conclusion holds for $n = 1$ and $n = 2$. The general case can be proved by induction. \square

Lemma 5.3. H in (4.5) is a polynomial in $\{w^{(\beta)}\}_{|\beta| \leq k+1}$ of weighted degree at most $\frac{(m+1)(m+2)}{2}$ with coefficients in $j^{k+1}F$.

Proof. Note that $|\alpha^i| \leq i$ for each $1 \leq i \leq m+1$. Thus $\mathcal{L}^{\alpha^1} f_1, \dots, \mathcal{L}^{\alpha^m} f_m, \mathcal{L}^{\alpha^{m+1}} g$ are all polynomials in $\{w^{(\beta)}\}_{1 \leq |\beta| \leq i}$ of weighted order at most i with coefficients in $j^i F$. Then the statement follows by an easy computation from its definition (4.4). \square

We now need to convert $H(z, w, \{\xi_\beta\}_{1 \leq |\beta| \leq k+1})$, where ξ_β corresponds to $w^{(\beta)}$, to $\tilde{H}(z, w, (\Phi_{ij}^{(\beta)})_{|\beta| \leq k-2})$. For that we study $\xi_\beta, |\beta| \geq 2$ as a polynomial in $\xi_1, \dots, \xi_n, \{\Phi_{ij}^{(\gamma)}\}_{0 \leq |\gamma| \leq |\beta|-2}$.

Recall

$$\xi_\beta = Q_\beta(\xi_1, \dots, \xi_n, \{\Phi_{ij}^{(\gamma)}\}_{0 \leq |\gamma| \leq |\beta|-2}), |\beta| \geq 2. \quad (5.1)$$

where Q_β is a polynomial in its argument.

Lemma 5.4. *Let $Q_\beta, |\beta| \geq 2$ be as above. Then Q_β is a polynomial of degree $\leq |\beta| - 1$ in its arguments.*

Proof. When $|\beta| = 2$, the statement is trivial since

$$\xi_{ij} = \Phi_{ij}, 1 \leq i, j \leq n.$$

The case $|\beta| = 3$ is verified as follows.

$$\begin{aligned} \xi_{ijk} &= (\Phi_{ij})_{z_k} + (\Phi_{ij})_w \xi_k + \sum_{l=1}^n (\Phi_{ij})_{\xi_l} \xi_{lk} \\ &= (\Phi_{ij})_{z_k} + (\Phi_{ij})_w \xi_k + \sum_{l=1}^n (\Phi_{ij})_{\xi_l} \Phi_{lk}, 1 \leq i, j, k \leq n. \end{aligned} \quad (5.2)$$

Then general case can be proved by induction. \square

Lemma 5.4 leads to the following lemma.

Lemma 5.5. $\tilde{H}(z, w, \xi, (\Phi_{ij}^{(\beta)})_{|\beta| \leq k-1})$ are polynomials of degrees $\leq \frac{(m+1)(m+2)}{2}$ in the arguments $\xi, \Phi_{ij}^{(\beta)}$.

Proof. Write any monomial of H in the following form:

$$h(s, t)w^{(\beta^1)} \dots w^{(\beta^\tau)},$$

with $|\beta^1| + \dots + |\beta^\tau| \leq \frac{m(m+1)}{2}$ by Lemma 5.3. Now each $w^{(\beta^i)}$, by Lemma 5.4, can be written as a polynomial $Q_{\beta^i}(\xi_1, \dots, \xi_n, \{\Phi_{ij}^{(\gamma)}\}_{0 \leq |\gamma| \leq |\beta^i|-2})$ of degree $\leq |\beta^i|$ if $|\beta^i| \geq 2$. The statement of Lemma 5.5 then follows easily. \square

Lemma 5.6. (1). *For each $l \geq 0$, there are $\binom{l+n-1}{n-1}$ distinct multiindices β such that $|\beta| = l$.*

Here we let $\binom{0}{0} = 1$. Consequently, $\{\xi, (\Phi_{ij}^{(\beta)})_{|\beta| \leq k-1}\}$ has

$$n + \frac{n(n+1)}{2} \binom{n+k-1}{n} \quad (5.3)$$

terms. Moreover, since $k \leq m - n + 1$, we have (5.3) bounded by

$$p(m, n) := n + \frac{n(n+1)}{2} \binom{m}{n}. \quad (5.4)$$

Proof. The proof of the lemma follows from elementary combinatorics. \square

We are now able, using elementary combinatorics again, to give an estimate for the number of terms in \tilde{H} and thus for the integer s . Note \tilde{H} is a polynomial of degree at most $(m+1)(m+2)/2$ with at most $p(m, n)$ variables.

Proposition 5.7. *One has*

$$s \leq \binom{(m+1)(m+2)/2 + p(m, n)}{p(m, n)} \quad (5.5)$$

Here $p(m, n)$ is defined by (5.4).

Combining now Proposition 5.7 with (4.16) and (4.22), we get

Theorem 5. *The integers $\nu(n, m)$ and $\mu(n, N)$ in Theorem 3 and Theorem 1 respectively can be explicitly chosen to be*

$$\nu(n, m) := 2 + m - n + \binom{(m+1)(m+2)/2 + p(m, n)}{p(m, n)} \quad (5.6)$$

$$\mu(n, N) := \nu(n, N). \quad (5.7)$$

Here $p(m, n)$ is the explicit expression given by (5.4).

Proof. Proposition 5.7 and formulas (4.16), (4.22) immediately imply (5.6). In order to prove (5.7), we note that the expression

$$\binom{(m+1)(m+2)/2 + p(m, n)}{p(m, n)}$$

is monotonous in m for fixed n . Indeed, if $m \leq m'$, set $c := (m+1)(m+2)/2$, $d := p(m, n)$, $c' := (m'+1)(m'+2)/2$, $d' := p(m', n)$. We have $c \leq c'$, $d \leq d'$. Then

$$\binom{c+d}{d} \leq \binom{c'+d}{d} = \binom{c'+d}{c'} \leq \binom{c'+d'}{c'} = \binom{c'+d'}{d'},$$

as required for the monotonicity. Now (5.7) follows from (4.23). \square

6. APPENDIX II

In this section, we provide a brief proof of Proposition 4.2. We first introduce following notions and definitions. Let \mathcal{M} be a n -dimensional (connected) complex manifold. Write $\Lambda_1, \dots, \Lambda_m$ as a basis of holomorphic vector field of \mathcal{M} . In particular, we assume $\Lambda_j h$, $1 \leq j \leq n$, is holomorphic whenever h is holomorphic. As before, for a multiple index $\alpha = (\alpha_1, \dots, \alpha_n)$, write $\Lambda^\alpha = \Lambda_1^{\alpha_1} \dots \Lambda_n^{\alpha_n}$. Let $H = (h_1, \dots, h_N)$ be a holomorphic map from \mathcal{M} to \mathbb{C}^N , $N \geq n$.

Definition 6.1. For each $l \geq 1, q \in \mathcal{M}$, define

$$E_l(q) := \text{Span}_{\mathbb{C}}\{\Lambda^\alpha H(q) : 1 \leq |\alpha| \leq l\}.$$

Remark 6.2. It is easy to see that if H is an embedding at q , then $\dim_{\mathbb{C}}(E_1(q)) = n$.

To establish Proposition 4.2, we need the following result.

Theorem 6. *Let O be an open subset of \mathcal{M} . Let $l \geq 1, n \leq m < N$. Assume that $\dim_{\mathbb{C}}(E_1(q)) = n, \dim_{\mathbb{C}}(E_l(q)) = \dim_{\mathbb{C}}(E_{l+1}(q)) = m$ for any $q \in O$. Then there exists complex numbers $\lambda_1, \dots, \lambda_N$ that are not all zero, such that,*

$$\lambda_1 \Lambda_j h_1 + \dots + \lambda_N \Lambda_j h_N = 0, \text{ for any } 1 \leq j \leq n.$$

Proof. The theorem basically follows from a similar argument as in [4] (See also [16]). We sketch a proof here. By the assumption that $\dim_{\mathbb{C}}(E_l(q)) = m$, shrinking O if necessary, there exist multi-indices $\alpha^1, \dots, \alpha^m$ with each $|\alpha^i| \leq l$ such that

$$\dim_{\mathbb{C}}\left(\text{Span}_{\mathbb{C}}\{\Lambda^{\alpha^1} H(q), \dots, \Lambda^{\alpha^m} H(q)\}\right) = m \text{ for every } q \in O. \quad (6.1)$$

Since $\dim_{\mathbb{C}}(E_1(q)) = n$, we can choose in (6.1) $\alpha^i = \epsilon^i, 1 \leq i \leq n$. Here we write for each $1 \leq i \leq n, \epsilon^i = (0, \dots, 0, 1, 0, \dots, 0)$, where the component “1” is at the i^{th} position, so that $\mathcal{L}^{\epsilon^i} = \mathcal{L}_i, 1 \leq i \leq n$.

The assumption that $\dim_{\mathbb{C}}(E_{l+1}(q)) = m$ implies that for any multiindex β with $|\beta| \leq l+1$.

$$\dim_{\mathbb{C}}\left(\text{Span}_{\mathbb{C}}\{\Lambda^{\alpha^1} H(q), \dots, \Lambda^{\alpha^m} H(q), \Lambda^\beta H(q)\}\right) = m \text{ for every } q \in O. \quad (6.2)$$

By equation (6.1), we conclude that there exists j_1, j_2, \dots, j_m such that, by shrinking O if necessary,

$$\begin{vmatrix} \Lambda^{\alpha^1} h_{j_1} & \dots & \Lambda^{\alpha^1} h_{j_m} \\ \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_{j_1} & \dots & \Lambda^{\alpha^m} h_{j_m} \end{vmatrix} \neq 0 \text{ at every } q \in O. \quad (6.3)$$

To make the notations simple, we assume, without loss of generality, that $j_1 = 1, \dots, j_m = m$. That is,

$$\begin{vmatrix} \Lambda^{\alpha^1} h_1 & \dots & \Lambda^{\alpha^1} h_m \\ \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_1 & \dots & \Lambda^{\alpha^m} h_m \end{vmatrix} \neq 0 \text{ at every } q \in O. \quad (6.4)$$

We conclude by equation (6.2) that for any multiindex β with $|\beta| \leq l+1$.

$$\begin{vmatrix} \Lambda^{\alpha^1} h_1 & \dots & \Lambda^{\alpha^1} h_m & \Lambda^{\alpha^1} h_{m+1} \\ \dots & \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_1 & \dots & \Lambda^{\alpha^m} h_m & \Lambda^{\alpha^m} h_{m+1} \\ \Lambda^\beta h_1 & \dots & \Lambda^\beta h_m & \Lambda^\beta h_{m+1} \end{vmatrix} \equiv 0 \text{ for every } q \in O. \quad (6.5)$$

Lemma 6.3. *For any $1 \leq \nu \leq n$, and $i_1 < i_2 < \dots < i_{m-1}$ with $\{i_1, \dots, i_{m-1}\} \subset \{1, 2, \dots, m\}$, the following holds:*

$$\Lambda_\nu \left(\frac{\begin{vmatrix} \Lambda^{\alpha^1} h_{i_1} & \dots & \Lambda^{\alpha^1} h_{i_{m-1}} & \Lambda^{\alpha^1} h_{m+1} \\ \dots & \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_{i_1} & \dots & \Lambda^{\alpha^m} h_{i_{m-1}} & \Lambda^{\alpha^m} h_{m+1} \end{vmatrix}}{\begin{vmatrix} \Lambda^{\alpha^1} h_1 & \dots & \Lambda^{\alpha^1} h_{m-1} & \Lambda^{\alpha^1} h_m \\ \dots & \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_1 & \dots & \Lambda^{\alpha^m} h_{m-1} & \Lambda^{\alpha^m} h_m \end{vmatrix}} \right) \equiv 0. \quad (6.6)$$

Proof. The conclusion follows from (6.5). Indeed, the numerator of the left hand side of (6.6) can be written as a summation of terms that are multiples of the left hand side of (6.5) for certain choices of β . A detailed proof can be copied from page 1391 of [4]. \square

Lemma 6.3 implies that the function in the big parentheses in the equation (6.6) is a constant in O . We now fix some notations. If $i_1 < \dots < i_{m-1}$ and $(i_1, \dots, i_{m-1}) = (1, 2, \dots, \hat{i}_0, \dots, m)$ (Here $(1, 2, \dots, \hat{i}_0, \dots, m)$ means $(1, 2, \dots, m)$ with the component “ i_0 ” missing.), then we write the constant

$$c_{i_0} := \frac{\begin{vmatrix} \Lambda^{\alpha^1} h_{i_1} & \dots & \Lambda^{\alpha^1} h_{i_{m-1}} & \Lambda^{\alpha^1} h_{m+1} \\ \dots & \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_{i_1} & \dots & \Lambda^{\alpha^m} h_{i_{m-1}} & \Lambda^{\alpha^m} h_{m+1} \end{vmatrix}}{\begin{vmatrix} \Lambda^{\alpha^1} h_{i_1} & \dots & \Lambda^{\alpha^1} h_{i_{m-1}} & \Lambda^{\alpha^1} h_{i_0} \\ \dots & \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_{i_1} & \dots & \Lambda^{\alpha^m} h_{i_{m-1}} & \Lambda^{\alpha^m} h_{i_0} \end{vmatrix}}$$

Consequently,

$$\begin{vmatrix} \Lambda^{\alpha^1} h_{i_1} & \dots & \Lambda^{\alpha^1} h_{i_{m-1}} & \Lambda^{\alpha^1} (h_{m+1} - c_{i_0} h_{i_0}) \\ \dots & \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_{i_1} & \dots & \Lambda^{\alpha^m} h_{i_{m-1}} & \Lambda^{\alpha^m} (h_{m+1} - c_{i_0} h_{i_0}) \end{vmatrix} \equiv 0 \text{ in } O. \quad (6.7)$$

Since i_0 may vary from 1 to m , we thus have m constants: c_1, \dots, c_m . We now prove the following lemma.

Lemma 6.4. *The following holds in O for any $i_1 < i_2 < \dots < i_{m-1}$ with $\{i_1, \dots, i_{m-1}\} \subset \{1, 2, \dots, m\}$:*

$$\begin{vmatrix} \Lambda^{\alpha^1} h_{i_1} & \dots & \Lambda^{\alpha^1} h_{i_{m-1}} & \Lambda^{\alpha^1} (h_{m+1} - \sum_{i=1}^m c_i h_i) \\ \dots & \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_{i_1} & \dots & \Lambda^{\alpha^m} h_{i_{m-1}} & \Lambda^{\alpha^m} (h_{m+1} - \sum_{i=1}^m c_i h_i) \end{vmatrix} \equiv 0 \text{ in } O.$$

Proof. Assume that $(i_1, \dots, i_{m-1}) = (1, 2, \dots, \hat{i}_0, \dots, m)$. Note that if $i \neq i_0$, i.e., $i \in \{i_1, \dots, i_{m-1}\}$, then

$$\begin{vmatrix} \Lambda^{\alpha^1} h_{i_1} & \dots & \Lambda^{\alpha^1} h_{i_{m-1}} & \Lambda^{\alpha^1} (c_i h_i) \\ \dots & \dots & \dots & \dots \\ \Lambda^{\alpha^m} h_{i_1} & \dots & \Lambda^{\alpha^m} h_{i_{m-1}} & \Lambda^{\alpha^m} (c_i h_i) \end{vmatrix} \equiv 0. \quad (6.8)$$

Indeed, the last column of the above matrix is just a constant multiple of one of the first $m-1$ columns. Then Lemma 6.4 follows easily from equations (6.7) and (6.8). \square

We recall the following lemma from [4].

Lemma 6.5. *Let $\mathbf{b}_1, \dots, \mathbf{b}_n$ and \mathbf{a} be n -dimensional column vectors with elements in \mathbb{C} , and let $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ denote the $n \times n$ matrix. Assume that $\det B \neq 0$, and that $\det(\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, \dots, \mathbf{b}_{i_{n-1}}, \mathbf{a}) = 0$ for any $1 \leq i_1 < i_2 < \dots < i_{n-1} \leq n$. Then $\mathbf{a} = \mathbf{0}$, where $\mathbf{0}$ is the n -dimensional zero column vector.*

By Lemmas 6.4, 6.5, and equation (6.4), we conclude that

$$\Lambda^{\alpha^j}(h_{m+1} - \sum_{i=1}^m c_i h_i) = 0, \quad \forall 1 \leq j \leq m.$$

In particular, when $1 \leq j \leq n$, since $\Lambda^{\alpha^j} = \Lambda_j$, We thus conclude that

$$\Lambda_j h_{m+1} - \sum_{i=1}^m c_i \Lambda_j h_i = 0, \quad 1 \leq j \leq n.$$

This establishes Theorem 6. □

Remark 6.6. We state the following fact which is an immediate consequence of Theorem 6. With the notions in Definition 6.1, assume that there do not exist constants $\lambda_1, \dots, \lambda_N$ that are not all zero such that

$$\lambda_1 \Lambda_j h_1 + \dots + \lambda_N \Lambda_j h_N = 0.$$

Let $l \geq 1, O$ an open subset of \mathcal{M} . Assume $\dim_{\mathbb{C}}(E_1(q)) = n, \dim_{\mathbb{C}}(E_l(q)) = m < N$, for any $q \in O$. Then for a generic $\tilde{q} \in O$, we have $E_l(\tilde{q}) \subsetneq E_{l+1}(\tilde{q})$.

Remark 6.6 implies Proposition 4.2.

REFERENCES

- [1] M. S. Baouendi, P. Ebenfelt, L. P. Rothschild. *Real Submanifolds in Complex Space and Their Mappings*, Princeton University Press, Princeton Math. Ser. **47**, Princeton, NJ, 1999.
- [2] S. Baouendi and X. Huang. *Super-rigidity for holomorphic mappings between hyperquadrics with positive signature*, J. Differential Geom. **69** (2005), no. 2, 379-398.
- [3] S. Baouendi, P. Ebenfelt, and X. Huang. *Super-rigidity for CR embedding of real hypersurfaces into hyperquadrics*, Adv. Math. **219** (2008), no. 5, 1427-1445.
- [4] S. Berhanu and M. Xiao, *On the C^∞ version of the reflection principle for mappings between CR manifolds*, Amer. J. Math. **137** (2015), 1365-1400.
- [5] E. Cartan. "Sur la geometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes II", Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) **1** (1932), no. 4, 333-354.
- [6] E. Cartan. *Sur les variétés á connexion projective*, Bull. Soc. Math. France **52** (1924), 205-241.
- [7] S. Chern and J. Moser. *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), no. 1, 219-271.
- [8] J. D'Angelo, *Hermitian analogues of Hilbert's 17-th problem*, Adv. Math. **226** (2011), no. 5, 4607-4637.
- [9] J. D'Angelo, *CR complexity and hyperquadric maps*, Analysis and Geometry, Springer Proceedings in Mathematics & Statistics **127**, (2015), 17-34.
- [10] J. D'Angelo and J. Lebl, *Hermitian symmetric polynomials and CR complexity*, J. Geom. Anal. **21** (2011), no. 3, 599-619.
- [11] P. Ebenfelt, X. Huang, D. Zaitsev. *Rigidity of CR-immersions into spheres*, Comm. Anal. Geom. **12** (2004), no. 3, 631-670.

- [12] P. Ebenfelt, X. Huang and D. Zaitsev. *The equivalence problem and rigidity for hypersurfaces embedded into hyperquadrics*, Amer. J. Math. **127** (2005), no. 1, 169-191.
- [13] P. Ebenfelt and B. Lamel. *Finite jet determination of CR embeddings*, J. Geom. Anal. **14** (2004), no. 2, 241-265.
- [14] P. Ebenfelt and R. Shroff. *Partial rigidity of CR embeddings of real hypersurfaces into hyperquadrics with small signature difference*, Comm. Anal. Geom. **23** (2015), no. 1, 159-190.
- [15] P. Ebenfelt and D. Son, *On the existence of holomorphic embeddings of strictly pseudoconvex algebraic hypersurfaces into spheres*, May, 2012 (arxiv: 1205.1237).
- [16] H. Fang, X. Huang, and M. Xiao, *Volume-preserving mappings between Hermitian symmetric spaces of compact type*, submitted, arxiv: 1602.01900.
- [17] J. J. V. Faran, *The nonembeddability of real hypersurface in spheres*, Proc. Amer. Math. Soc. **103**, (1988), no. 3, 902-904.
- [18] F. Forstneric, *Embedding strictly pseudoconvex domains into balls*, Transactions of AMS, **295** (1986), no. 1, 347-368.
- [19] F. Forstneric, *Most real-analytic Cauchy-Riemann manifolds are nonalgebraizable*, Manuscripta Math. **115** (2004), no. 4, 489-494.
- [20] X. Huang, X. Li, and M. Xiao, *Non-embeddability into a fixed sphere for a family of compact real algebraic hypersurfaces*, Int. Math. Res. Not.(IMRN), DOI: 10.1093/imrn/rnu167.
- [21] X. Huang and D. Zaitsev, *Non-embeddable real algebraic hypersurface*, Math. Z., **275** (2013), no. 3-4, 657-671.
- [22] X. Huang and M. Xiao, *Chern-Moser-Weyl tensor and embedding into hyperquadrics*, submitted, arxiv: 1606.09145.
- [23] X. Huang and Y. Zhang, *Monotonicity for Chern-Moser-Weyl curvature tensor and CR embeddings*, Science in China Series A: Mathematics, **52** Dec. 2009, no. 12, 2617-2627.
- [24] I. Kossovskiy and B. Lamel. *On the Analyticity of CR-diffeomorphisms*. To appear in the American Journal of Math. (AJM). Available at <http://arxiv.org/abs/1408.6711>
- [25] I. Kossovskiy and B. Lamel. *New extension phenomena for solutions of tangential Cauchy-Riemann equations*, Comm. Partial Differential Equations **41** (2016), no. 6, 925-951.
- [26] I. Kossovskiy and R. Shafikov. *Divergent CR-equivalences and meromorphic differential equations*, 2014. To appear in J. Europ. Math. Soc. (JEMS). Available at <http://arxiv.org/abs/1309.6799>.
- [27] I. Kossovskiy and R. Shafikov. *Analytic Differential Equations and Spherical Real Hypersurfaces*. J. Diff. Geom. (JDG), Vol. 102, No. 1 (2016), pp. 67 - 126.
- [28] B. Lamel, *A reflection principle for real-analytic submanifolds of complex spaces*, J. Geom. Anal., **11** (2001), no. 4, 625-631.
- [29] B. Lamel, *A C^∞ -regularity for nondegenerate CR mappings*, Monatsh. Math., **142** (2004), no. 4, 315-326.
- [30] J. Merker. *Nonrigid spherical real analytic hypersurfaces in \mathbb{C}^2* . Complex Var. Elliptic Equ. **55** (2010), no. 12, 1155-1182.
- [31] A. Ostrowski. *Über Dirichletsche Reihen und algebraische Differentialgleichungen*, (German) Math. Z. **8** (1920), no. 3-4, 241-298.
- [32] B. Segre. *Questioni geometriche legate colla teoria delle funzioni di due variabili complesse*, Rendiconti del Seminario di Matematici di Roma, II, Ser. 7 (1932), no. 2, 59-107.
- [33] A. Sukhov. *Segre varieties and Lie symmetries*, Math. Z. **238** (2001), no. 3, 483-492.
- [34] A. Sukhov. *On transformations of analytic CR-structures*, Izv. Math. **67** (2003), no. 2, 303-332.
- [35] A. Tresse. *Détermination des invariants ponctuels de l'équation différentielle du second ordre $y'' = \omega(x, y, y')$* , Hirzel, Leipzig, 1896.
- [36] K. Wolsson, *Linear dependence of a function set of m variables with vanishing generalized Wronskians*, Linear Algebra and its applications, **117** (1989), 73-80.
- [37] S. Webster. *On the mappings problem for algebraic real hypersurfaces*, Invent. Math., **43** (1977), no. 1, 53-68.

- [38] D. Zaitsev, *Obstructions to embeddability into hyperquadrics and explicit examples*, *Math. Ann.*, **342** (2008), no. 3, 695-726.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MASARYK UNIVERSITY, BRNO, CZECHIA/
FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, AUSTRIA
E-mail address: `kossovskiy@math.muni.cz`, `ilya.kossovskiy@univie.ac.at`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
E-mail address: `mingxiao@illinois.edu`