Sec 9.3 

e.g. Find a formal Fourier series solution of the endpoint value problem

\[ x'' + 2x = t, \quad x(0) = x'(\pi) = 0 \]

**Step 1.** Determine the period by the endpoint.

Since the endpoints are 0, \( \pi \), thus the period \( L = \pi \).

**Step 2.** \( f(t) = t, \quad 0 < t < L \)

We need to write \( f(t) \) as a Fourier series. We choose from cosine series or sine series.

Since we would like \( x(t) \) automatically satisfy \( x(0) = x'(\pi) = 0 \).

So we will write \( x(t) \) as a cosine series

\[ x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} \]
Then \( x''(t) + 2x'(t) \) will only have cosine terms. \( \Rightarrow \)
\( x''(t) + 2x(t) = f(t) \) will only have cosine terms. Hence we write \( f(t) \) as cosine terms.

We compute that
\[
\int_0^t f(s) \, ds = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}.
\]

Write
\[
x(t) = \frac{x_0}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} \cos \frac{n\pi}{2} t
\]
\[
= \frac{x_0}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} a_n \cos nt.
\]
Then

\[ x''(t) + 2x(t) = -\sum_{n=1}^{\infty} \frac{\pi^2}{n^2} \cos nt + c_0 + 2 \sum_{n=1}^{\infty} \frac{\pi}{n} A_n \cos nt \]

\[ = c_0 + \sum_{n=1}^{\infty} \left( \frac{2an - n^2an}{n^2} \right) \cos nt \]

\[ x''(t) + 2x(t) = f(t) \text{ yields that} \]

\[ c_0 + \sum_{n=1}^{\infty} \left( \frac{2an - n^2an}{n^2} \right) \cos nt = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\cos nt}{n^2} \]

Then we have

\[ c_0 = \frac{\pi}{2} \]

\[ 2an - n^2an = \frac{\pi}{2} - \frac{4}{\pi} \frac{1}{n^2n} \text{, } n \text{ odd} \]

\[ 2an - n^2an = 0 \text{, } n \text{ even} \]

\[ \Rightarrow c_0 = \frac{\pi}{2} \]

\[ An = 0 \text{, } n \text{ even} \]

\[ An = \frac{4}{\pi n^2(n^2 - 2)} \text{, } n \text{ odd} \]
Hence the formal series \( \sin \theta \) is

\[
x(t) = \frac{\pi}{4} + \frac{4}{\pi} \sum_{n \neq 0} \frac{\cos n t}{n^2 (n^2 - 2)}
\]
Thm. Termwise integration of Fourier Series.
Suppose that \( f \) is a piecewise continuous periodic function with period \( 2L \) and Fourier Series
\[
  f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)
\]
which may not converge. Then
\[
  \int_{-L}^{L} f(t) \, dt = \frac{a_0 t}{2} + \sum_{n=1}^{\infty} \frac{L}{n\pi} \left[ a_n \sin \frac{n\pi}{L} - b_n \left( \cos \frac{n\pi t}{L} - 1 \right) \right]
\]
with the series on the RHS convergent for all \( t \). Note that the above series is obtained by integrating the Fourier series of \( f \). If \( a_0 \neq 0 \), it is not a Fourier series because of its linear term \( t \).
Let $f(t)$ be a periodic function with period $2\pi$. Then

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

We first compute the Fourier Series of $f$,

$$f(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

By the theorem above, we integrate to get

$$F(t) = \int_{0}^{t} f(s) \, ds$$

$$= \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} \int_{0}^{t} \frac{1}{n} \sin ns \, ds$$

$$= \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n^2} (1 - \cos nt)$$

$$= \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n^2} - \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\cos nt}{n^2}.$$
On the other hand, we can compute $P_f$

$$F(t) = \int_0^t f(s) \, ds = |t|, \quad -\pi < t < \pi.$$ 

Q. Check the Fourier series of

$$F(t) = |t|, \quad -\pi < t < \pi \text{ with period } 2\pi$$

is

$$\frac{4}{\pi} \frac{2}{n^2} = -\frac{4}{\pi} \frac{2}{n^2} \frac{\cos nt}{h^2}$$

$$\text{Hint:} \quad \frac{2}{n^2} = \left( \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right)$$

$$= \frac{\pi^2}{8}$$