On the C^∞ version of the reflection principle for mappings between CR manifolds

S. Berhanu *
Department of Mathematics
Temple University
Philadelphia, PA 19122

Ming Xiao
Department of Mathematics
Rutgers University
New Brunswick, NJ 08854

October 14, 2014

Abstract

We prove a smooth version of the classical Schwarz reflection principle for CR mappings between an abstract CR manifold M and a generic CR manifold embedded in euclidean complex space. As a consequence of our results, we settle a conjecture of X. Huang formulated in [Hu2].

1 Introduction

In this paper we study the regularity problem for CR mappings between CR manifolds where the CR dimension of the source manifold is less than or equal to that of the target manifold. Our results imply a positive answer to a conjecture of X. Huang in [Hu2] and provide a solution to a question raised in [Fr1] (see Corollaries 2.10 and 2.11).

One of our theorems can be viewed as a smooth version of the analyticity theorems of Forstneric ([Fr1]) and Huang [Hu1-2] for CR mappings between CR manifolds of differing dimensions. The article is devoted to results along the line of research on establishing the smooth version of the Schwarz reflection principle for holomorphic maps in several variables. Results of this type were first proved in the 70's starting with the work of Fefferman [Fe], Lewy [Le] and Pinchuk [Pi]. The seminal work [BJT] has influenced a lot of work on the subject. For

*Work supported in part by NSF DMS 1300026

extensive surveys and many references on this research, the reader may consult the articles by Bedford [Be], Forstneric [Fr2], and Bell-Narasimhan [BN]. Among the many related papers we mention here [CKS], [CGS], [CS], [DW], [EH], [EL], [Fr1], [Fr3], [Hu1], [Hu2], [KP], [K], [La1], [La2], [La3], [M], [NWY], [Tu], and [W]. In [Fr3] Forstneric generalized Fefferman's theorem to CR homeomorphisms $f : M \to M'$ where f^{-1} is CR, M and M' are generic CR submanifolds of $\mathbb{C}^n$ with the same CR dimension. The book [BER] by Baouendi, Ebenfelt, and Rothschild contains a detailed account and references related to the study when the manifolds are real analytic or real algebraic.

We prove results on the smoothness of CR maps where the source manifold M is assumed to be an abstract (not necessarily embeddable) CR manifold. We mention that the results are new even when M is embeddable. Our first main result, Theorem 2.3, generalizes to abstract CR manifolds a theorem of Lamel in [La1] proved for generic CR manifolds embedded in complex spaces. The second main result, Theorem 2.5, establishes the smoothness on a dense open subset of a C^k CR mapping $F : (M, \mathcal{V}) \to (M', \mathcal{V}')$ where $(M, \mathcal{V})$ is an abstract CR manifold of CR dimension n and $M' \subset \mathbb{C}^{n+k}$ is a hypersurface that is strongly pseudoconvex. A condition on the Levi form of $(M, \mathcal{V})$ is assumed in Theorem 2.5.

Our approach is based on the framework established by Roberts [GR] in his thesis and a later paper by Lamel in [La1]. The notion of k_0-nondegeneracy of a CR mapping (Definition 2.1) and the "almost holomorphic" implicit function theorem of Lamel in [La1] and [La2] play crucial roles in the proofs and formulations of our results. The proof is also motivated by the study of the real analyticity for CR maps between real analytic strongly pseudoconvex hypersurfaces of different dimensions in Forstneric [Fr1] and Huang [Hu1]. We mention that in [Fr1], Forstneric conjectured that F must be real analytic when $M_1 \subset \mathbb{C}^{n+1}$ and $M_2 \subset \mathbb{C}^{n+k}$ $(k \geq 2, n \geq 1)$ are real analytic hypersurfaces with M_1 of finite type, M_2 strongly pseudoconvex, and he proved that this is indeed the case on a dense open set when F is smooth. The conjecture of Forstneric was settled by Huang ([Hu1]) who obtained the analyticity of F on a dense open subset assuming only that $F \in C^k$. The analyticity of F, when both M_1 and M_2 are as in [Fr1] and when F is only C^k-smooth also follows from Theorem 2.5 in this paper and Forstneric's analyticity result when F is smooth.

The paper is organized as follows. In Section 2 we state the main results and prove a preliminary microlocal regularity result that is used in the proof of Theorem 2.5 and supplies a good class of examples to which Theorem 2.3 can be applied. Section 3 contains the proof of Theorem 2.3 and Theorem 2.5 is proved in Section 4. In an appendix we indicate why we focus only on CR mappings where the target manifold has a higher CR dimension than that of the source manifold.

Acknowledgement: We use this opportunity to thank the anonymous referee for his helpful comments that have resulted in an improvement of the presentation of the results in this paper. The second author expresses his gratefulness to his advisor Xiaojun Huang for his constant help and encouragement. He also thanks Xiaoshan Li for his interest and help in this work.

2 Statements of the results and the proof of a preliminary result

Let M be a smooth manifold and let $\mathcal{V}$ be a subbundle of $\mathbb{C}TM$, the complexified tangent bundle of M. The pair $(M,\mathcal{V})$ is called an abstract CR manifold if $\mathcal{V}$ is involutive and if for each $p\in M$, $\mathcal{V}_p\cap\overline{\mathcal{V}}_p=0$. Recall that the involutivity of $\mathcal{V}$ means that the space of smooth sections of $\mathcal{V}, C^\infty(M,\mathcal{V})$, is closed under commutators. Let n be the complex dimension of the fibers $\mathcal{V}_p$ and write $\dim_{\mathbb{R}}M=2n+d$. The number n is called the CR dimension of M, and d is called the CR codimension of M. If $d=1$, the CR manifold is said to be of hypersurface type. A smooth section of $\mathcal{V}$ is called a CR vector field and a function (or distribution) is called CR if $Lf=0$ for any CR vector field L. The CR manifold $(M,\mathcal{V})$ is called locally embeddable if for any $p_0\in M$, there exist m complex-valued C^∞ functions $Z_1,\cdots,Z_m$ defined near p_0 with $m=\dim_{\mathbb{R}}M-n,$ such that the Z_j are CR functions near p_0, and the differentials $dZ_1,\cdots,dZ_m$ are $\mathbb{C}-$linearly independent. In this case, the mapping

$$p\mapsto Z(p)=(Z_1(p),\cdots,Z_m(p))\in\mathbb{C}^m=\mathbb{C}^{n+d}$$

is an immersion near p_0. Thus, if U is a small neighborhood of p_0, then $Z(U)$ is an embedded submanifold of $\mathbb{C}^m$ and is a generic CR submanifold of $\mathbb{C}^m$ whose induced CR bundle agrees with the push forward $Z_*(\mathcal{V})$ (see [BER] and [J] for more details). Let $(M',\mathcal{V}')$ be another abstract CR manifold with CR dimension n' and CR codimension d'. A CR mapping of class $C^k(k\geq 1)$ $H:(M,\mathcal{V})\to(M',\mathcal{V}')$ is a C^k mapping $H:M\to M'$ such that for each $p\in M$,

$$dH(\mathcal{V}_p)\subset\mathcal{V}'_{H(p)}.$$

When $(M',\mathcal{V}')$ is a generic CR submanifold of $\mathbb{C}^{N'}(N'=n'+d')$, then a C^k mapping $H=(H_1,\cdots,H_{N'}):M\to M'$ is a CR mapping if and only if each H_j is a CR function. One of our main results generalizes to an abstract CR manifold $(M,\mathcal{V})$ a regularity theorem of Lamel ([La1]) for CR mappings of embedded CR manifolds. We need to recall from [La1] the notion of nondegenerate CR mappings. Let $\widetilde{M}\subset\mathbb{C}^N$ and $\widetilde{M'}\subset\mathbb{C}^{N'}$ be two generic CR submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively. If d and d' denote the real codimensions of $\widetilde{M}$ and $\widetilde{M'}$, then $n=N-d$ and $n'=N'-d'$ are the CR dimensions of $\widetilde{M}$ and $\widetilde{M'}$ respectively. Let $H:\widetilde{M}\to\widetilde{M'}$ be a CR mapping of class C^k.

Definition 2.1. *([La1]) Let $\widetilde{M},\widetilde{M'}$ and H be as above and $p_0\in\widetilde{M}$. Let $\rho=(\rho_1,\cdots,\rho_{d'})$ be local defining functions for $\widetilde{M'}$ near $H(p_0)$, and choose a basis $L_1,\cdots,L_n$ of CR vector* *fields for* $\widetilde{M}$ *near* p_0*. If* $\alpha = (\alpha_1, \cdots, \alpha_n)$ *is a multiindex, write* $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$*. Define the increasing sequence of subspaces* $E_l(p_0)(0 \leq l \leq k)$ *of* $\mathbb{C}^{N'}$ *by*

$$E_l(p_0) = \mathrm{Span}_{\mathbb{C}}\{L^\alpha \rho_{\mu,Z'}(H(Z), \overline{H(Z)})|_{Z=p_0} : 0 \leq |\alpha| \leq l, 1 \leq \mu \leq d'\}.$$

Here $\rho_{\mu,Z'} = (\frac{\partial \rho_\mu}{\partial z'_1}, \cdots, \frac{\partial \rho_\mu}{\partial z'_{N'}})$*, and* $Z' = (z'_1, \cdots, z'_{N'})$ *are the coordinates in* $\mathbb{C}^{N'}$*. We say that* H *is* k_0-*nondegenerate at* p_0 $(0 \leq k_0 \leq k)$ *if*

$$E_{k_0-1}(p_0) \neq E_{k_0}(p_0) = \mathbb{C}^{N'}.$$

The dimension of $E_l(p)$ over $\mathbb{C}$ will be called the l^{th} geometric rank of F at p and it will be denoted by $\mathrm{rank}_l(F, p)$.

For the invariance of this definition under the choice of the defining functions ρ_μ, the basis of CR vector fields and the choice of holomorphic coordinates in $\mathbb{C}^{N'}$, the reader is referred to [La2]. An intrinsic definition was presented in the paper [EL]. If M is a manifold for which the identity map is k_0-nondegenerate, then the manifold is called k_0-*nondegenerate*. This latter notion was introduced for embedded hypersurfaces in [BHR] and it is shown in [E] that it can be formulated for an abstract CR manifold. The reader is referred to these two references for a detailed treatment of this concept and its connection with holomorphic nondegeneracy in the sense of Stanton ([S]). In particular, in [BHR] and [E] it is shown that Levi-nondegeneracy of a CR manifold is equivalent to $1-$nondegeneracy. Thus the notion of k_0-nondegeneracy of a CR manifold can be viewed as a generalization of Levi nondegeneracy.

The main result in [La1] is as follows:

Theorem 2.2. *Let* $M \subset \mathbb{C}^N, M' \subset \mathbb{C}^{N'}$ *be smooth generic submanifolds of* $\mathbb{C}^N$ *and* $\mathbb{C}^{N'}$ *respectively,* $p_0 \in M, H = (H_1, \cdots, H_{N'}) : M \to M'$ *a* C^{k_0} *CR map which is* k_0-*nondegenerate at* p_0 *and extends continuously to a holomorphic map in a wedge* W *with edge* M*. Then* H *is smooth in some neighborhood of* p_0*.*

Here recall that if $p_0 \in M$, $d =$ the CR codimension of M, and $U \subset \mathbb{C}^N$ is a neighborhood of p_0, a wedge W with edge M centered at p_0 is defined to be an open set of the form:

$$W = \{Z \in U : r(Z, \overline{Z}) \in \Gamma\},$$

where $\Gamma \subset \mathbb{R}^d$ is an open convex cone, and $r = (r_1, \cdots, r_d)$ are defining functions for M near p_0. In section 3, we will prove the following generalization of Theorem 2.2.

Theorem 2.3. *Let $(M, \mathcal{V})$ be an abstract CR manifold and $M' \subset \mathbb{C}^{N'}$ a generic CR submanifold of $\mathbb{C}^{N'}$. Let $H = (H_1, \cdots, H_{N'}) : M \to M'$ be a CR mapping of class C^{k_0} which is k_0-nondegenerate at p_0 and assume that for some open convex cone $\Gamma \subset \mathbb{R}^d$,*

$$\mathrm{WF}(H_j)|_{p_0} \subset \Gamma, j = 1, \cdots, N'$$

where d is the CR codimension of M. Then H is C^∞ in some neighborhood of p_0.

In Theorem 2.3, $\mathrm{WF}(u)$ denotes the C^∞ wave front set of u, that is,

$$\mathrm{WF}(u) = \{\sigma \in T^*M : u \text{ is not microlocally smooth at } \sigma\}.$$

For details about the C^∞ wave front set of a function, see [H].

Remark 2.4. *In Theorem 2.2, the assumption that H is the boundary value of a holomorphic function in a wedge implies the much weaker condition that $WF(H_j)|_{p_0} \subset \Gamma$ for some Γ as in Theorem 2.3. Indeed, in the embedded case as in Theorem 2.2, a CR function h on M is the boundary value of a holomorphic function in a wedge if and only if its hypo-analytic wave front set is contained in an acute cone which means that the FBI transform of h decays exponentially. Our assumption in Theorem 2.3 only requires the FBI transform to decay rapidly.*

In what follows, given a CR manifold $(M, \mathcal{V})$, T^0 will denote its characteristic bundle, that is, $T^0 = \{\sigma \in T^*M : \langle \sigma, L\rangle = 0 \text{ for every smooth section } L \text{ of } \mathcal{V}\}$.

Theorem 2.5. *Let $(M, \mathcal{V})$ be an abstract CR manifold with CR dimension $n \geq 1$ such that the Levi form at every covector $\sigma \in T^0$ has a nonzero eigenvalue. Suppose $M' \subset \mathbb{C}^{n+k}$ is a hypersurface that is strongly pseudoconvex ($k \geq 1$) and let $\mathcal{V}'$ denote the CR bundle of M'. Let $F = (F_1, \cdots, F_{n+k}) : M \to M'$ be a CR mapping of class C^k whose differential $dF : \mathcal{V}_p \to \mathcal{V}'_{F(p)}$ is injective at every $p \in M$. Then F is C^∞ on a dense open subset of M.*

We note that the preceding theorem allows a weakening of the smoothness assumption in Theorem 1.2 of [EL] on finite jet determination. The theorem also implies that some of the results in [BR] hold under a weaker smoothness assumption on the CR maps involved. If $M \subset \mathbb{C}^N, M' \subset \mathbb{C}^{N'}$ are hypersurfaces, with M Levi nondegenerate at $p \in M$ and $F : M \to M'$ is a CR mapping which is transversal at p, that is, $dF(\mathbb{C}T_pM)$ is not contained in $\mathcal{V}'_{F(p)} + \overline{\mathcal{V}'_{F(p)}}$, then F is a local embedding (see section 3.4 in [EL]). Many other situations where $(M, \mathcal{V})$ and $(M', \mathcal{V}')$ are as in Theorem 2.5 and dF is injective can be found in the work [BR].

Let M, M', F be as in Theorem 2.5. Define

$$\Omega_1 = \{p \in M : \mathrm{rank}_k(F, p) = n + k\},$$

$$\Omega_2 = \{p \in M : \mathrm{rank}_k(F, q) \leq n + k - 1 \text{ at all points } q \text{ in some neighborhood of } p \text{ in } M\},$$

$$\Omega = \{p \in M : F \text{ is smooth in a neighborhood of } p \text{ in } M\}.$$

For each $1 \leq l \leq k$, we also set

$$S_l := \{p \in M : \mathrm{rank}_l(F, p) \leq n + l - 1\}.$$

Note that $\Omega_1, \Omega_2, \Omega$ are all open in M, and $\Omega_1 \cup \Omega_2$ is dense in M. Moreover, $\Omega_2 \subset \overset{\circ}{S}_k$.

Definition 2.6. *Let M, M', F, S_l be as above. For any $p \in \Omega_2$, we define the degenerate degree of F at p to be*

$$\min\{1 \leq l \leq k : \textit{there exists a neighborhood } O \textit{ of } p \textit{ such that } O \subset S_l\},$$

and write it as $\deg(F, p)$.

Remark 2.7. *Definition 2.6 is independent of the choices of the defining function, the basis of CR vector fields and the choice of holomorphic coordinates in $\mathbb{C}^{n+k}$. For any $p \in \Omega_2$, by Lemma 4.1 in Section 4, $\mathrm{rank}_1(F, p) = n + 1$, which yields that $\deg(F, p) \geq 2$. Moreover, by the definition of $\deg(F, p)$, if we let $d = \deg(F, p)$, there exists a neighborhood $\widetilde{O}$ of p in M and $\{p_i\}_0^\infty \subset \widetilde{O}$ such that $\mathrm{rank}_d(F, q) \leq n + d - 1$ for all $q \in \widetilde{O}$, $\{p_i\}$ converges to p, and $\mathrm{rank}_{d-1}(F, p_i) = n + d - 1$ for all $i \geq 0$.*

Theorem 2.5 will follow from the following Theorem and Theorem 2.9 below which together with Theorem 2.3 imply that F is smooth in Ω_1, that is, $\Omega_1 \subset \Omega$.

Theorem 2.8. *For any $p \in \Omega_2$, there exists a sequence $\{p_i\}_{i=0}^\infty \subset \Omega$ that converges to p.*

It follows that Ω is dense in $\Omega_1 \cup \Omega_2$, and hence in M. We remark that Theorem 4.8 will show that if for some integer l, $1 \leq l \leq k - 1$, $\mathrm{rank}_{l+1}(F, q) = n + l$ for all points q in a neighborhood of p, and $\mathrm{rank}_l(F, p) = n + l$, then $F : M \to M'$ is smooth in a neighborhood of p, where F, M, and M' are as in Theorem 2.5. That is, such points p are in Ω.

Before we present the proofs of Theorem 2.3 and Theorem 2.5, we will prove the following result which supplies a class of examples to which Theorem 2.3 applies. This theorem will also be used in the proof of Theorem 2.5. The result may be viewed as the smooth version of Hans Lewy's extendability theorem in the embedded case.

Theorem 2.9. *Let $(M, \mathcal{V})$ be an abstract CR manifold, $\sigma \in T_p^0$, with the property that the Levi form at σ has a negative eigenvalue. Then if u is a CR function (or distribution) near $p, \sigma \notin \mathrm{WF}(u)$. In particular, if the Levi form at every covector $\eta \in T_p^0$ has a nonzero eigenvalue, then there is an open convex cone $\Gamma \subset \mathbb{R}^d (d =$ the CR codimension of $M)$ such that for every CR function u near p, $\mathrm{WF}(u)|_p \subset \Gamma$.*

Theorem 2.5 implies the following corollary which settles Huang's conjecture in [Hu2]:

Corollary 2.10. *Let $M \subset \mathbb{C}^{n+1}$, $M' \subset \mathbb{C}^{n+k}$ be smooth strongly pseudoconvex real hypersurfaces with $n \geq 1, k \geq 1$. Let $F : M \to M'$ be a CR mapping of class C^k. Then $F \in C^\infty(\Omega)$ on a dense open subset $\Omega \subset M$.*

Theorem 2.5 also provides a solution to a question of Forstneric in [Fr1] using methods different from the ones employed by Huang in the solution that he gave in [Hu1]:

Corollary 2.11. *Let $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ be real analytic hypersurfaces $(1 < N < N')$, M of finite type (in D'Angelo's sense) and M' strongly pseudoconvex. If $F : M \to M'$ is a CR mapping of class $C^{N'-N+1}$, then F extends to a holomorphic map on a neighborhood of an open, dense subset of M.*

Proof. Let $p \in M$. If every neighborhood of p contains a point where the Levi form has a positive and a negative eigenvalue, then p is in the closure of the set where F is smooth. We may therefore assume that a neighborhood D of p is pseudoconvex. Note next that since M doesn't contain a complex variety of positive dimension, it can not be Levi flat in any neighborhood of p. We can therefore assume that p is in the closure of the set of strictly pseudoconvex points in M. This latter assertion can be seen by using the arguments in Lemma 6.2 in [BHR]. In that paper, M was assumed algebraic but the reasoning in the Lemma is valid for M as in this corollary. Since we may assume that F is non constant, at a point of strict pseudo convexity, the differential dF is injective. The corollary now follows from Theorem 2.5 and the analyticity theorem in [Fr1]. □

In [Hu1] M was assumed strongly pseudoconvex. However, as indicated above, when M is of finite type in D'Angelo's sense, the problem is reduced to the strongly pseudoconvex case.

We now present the proof of Theorem 2.9.

Proof. Recall that the Levi form of $(M, \mathcal{V})$ at the characteristic covector $\sigma \in T_p^0$ is the hermitian form on $\mathcal{V}$ defined by

$$\mathcal{L}_\sigma(v, w) = \frac{1}{2\sqrt{-1}} \langle \sigma, [L, \overline{L}']_p \rangle,$$

where L and L' are smooth sections of $\mathcal{V}$ defined near p with $L(p) = v, L'(p) = w$. When this form has a negative eigenvalue, there is a CR vector L near p such that

$$\frac{1}{2\sqrt{-1}}\langle \sigma, [L, \overline{L}]_p\rangle < 0.$$

We may therefore assume that we are in coordinates $(x, t) \in \mathbb{R}^{n_0} \times \mathbb{R}$ that vanish at p,

$$L = \frac{\partial}{\partial t} + \sqrt{-1}\sum_{j=1}^{n_0} b_j(x, t)\frac{\partial}{\partial x_j},$$

where the b_j are C^∞ and real-valued functions near $(0, 0)$, $\sigma = (0, 0, \xi^0, 0)$ satisfies $b(0, 0)\cdot\xi^0 = 0, (b = (b_1, \cdots, b_{n_0}))$ and

$$\left\langle (\xi^0, 0), \frac{[L, \overline{L}]_0}{2\sqrt{-1}} \right\rangle = -\frac{\partial b}{\partial t}(0)\cdot \xi^0 < 0. \tag{2.1}$$

Assume that $Lu = 0$ near $(0, 0)$. We wish to show that $\sigma \notin \mathrm{WF}(u)$.

We introduce an additional variable $s \in \mathbb{R}$ and define

$$L_1 = \frac{\partial}{\partial s} + \sqrt{-1}L = \frac{\partial}{\partial s} + \sqrt{-1}\frac{\partial}{\partial t} - \sum_{j=1}^{n_0} b_j(x, t)\frac{\partial}{\partial x_j}.$$

Let $Z_i(x, t, s)\,(1 \le i \le n_0)$ be C^∞ functions near the origin satisfying

$$L_1 Z_i(x, t, s) = O(s^l),\ \text{as } s \to 0,\ \forall\ l \ge 1, l \in \mathbb{N},\ \text{and } Z_i(x, t, 0) = x_i.$$

Set $Z_{n_0+1}(x, t, s) = t - \sqrt{-1}s$. For $1 \le i \le n_0$, we can write $Z_i(x, t, s) = x_i + s\psi_i(x, t, s)$ for some C^∞ functions ψ_i. We have, for any $l \ge 1, 1 \le i \le n_0$,

$$s\frac{\partial\psi_i}{\partial s}(x, t, s) + \psi_i(x, t, s) + \sqrt{-1}s\frac{\partial\psi_i}{\partial t}(x, t, s) - \sum_{j=1}^{n_0} b_j(x, t)\left(\delta_{ij} + s\frac{\psi_i}{\partial x_j}(x, t, s)\right) = O(s^l). \tag{2.2}$$

It follows that

$$\psi_i(x, t, 0) = b_i(x, t),\ 1 \le i \le n_0. \tag{2.3}$$

Differentiating equation (2.2) with respect to s leads to,

$$s\frac{\partial^2\psi_i}{\partial s^2} + 2\frac{\psi_i}{\partial s} + \sqrt{-1}\frac{\psi_i}{\partial t} + \sqrt{-1}s\frac{\partial^2\psi_i}{\partial s\partial t} - \sum_{j=1}^{n_0} b_j\frac{\partial\psi_i}{\partial x_j} - s\sum_{j=1}^{n_0} b_j\frac{\partial^2\psi_i}{\partial s\partial x_j} = O(s^l),\ \forall\ l \ge 1.$$

Evaluating the latter at $s = 0$, we get, for any $1 \leq i \leq n_0$,

$$2\frac{\partial \psi_i}{\partial s}(x,t,0) + \sqrt{-1}\frac{\partial \psi_i}{\partial t}(x,t,0) - \sum_{j=1}^{n_0} b_j(x,t)\frac{\partial \psi_i}{\partial x_j}(x,t,0) = 0,$$

which together with equation (2.3) leads to:

$$\mathrm{Im}\,\psi_i(x,t,0) = 0 \text{ and } \frac{\partial}{\partial s}\mathrm{Im}\,\psi_i(x,t,0) = -\frac{1}{2}\frac{\partial b_i}{\partial t}(x,t),\ \forall\ 1 \leq i \leq n_0. \tag{2.4}$$

We will use the FBI transform in (x,t) space. For the solution $u = u(x,t)$, at level $s = s'$, we write,

$$\mathcal{F}(x,t,\xi,\tau,s') = \int_{\mathbb{R}^{n_0+1}} e^{Q(x,t,x',t',\xi,\tau,s')}\eta(x',t')u(x',t')dZ_1(x',t',s') \wedge \cdots \wedge dZ_{n_0+1}(x',t',s'),$$

where $(\xi,\tau) \in \mathbb{R}^{n_0} \times \mathbb{R}, \eta \in C_0^\infty(\mathbb{R}^{n_0+1}), \eta(x,t) \equiv 1$ for $|x|^2 + t^2 \leq r^2, \eta(x,t) \equiv 0$ when $|x|^2 + t^2 \geq 2r^2$ for some $r > 0$ to be fixed. Here

$$\begin{aligned} Q(x,t,x',t',\xi,\tau,s') = {} & \sqrt{-1}\langle(\xi,\tau),(x - Z(x',t',s'), t - Z_{n_0+1}(x',t',s'))\rangle \\ & - K|(\xi,\tau)|((x - Z(x',t',s'))^2 + (t - Z_{n_0+1}(x',t',s'))^2), \end{aligned}$$

where $Z = (Z_1,\cdots,Z_{n_0}), (x - Z(x',t',s'))^2 = \sum_{j=1}^{n_0}(x_j - Z_j(x',t',s'))^2$, and K is a positive number which will be determined.

Let $M_i = \sum_{j=1}^{n_0} a_{ij}(x,t,s)\frac{\partial}{\partial x_j}, 1 \leq i \leq n_0$ and $M_{n_0+1} = \frac{\partial}{\partial t} + \sum_{j=1}^{n_0} c_j(x,t,s)\frac{\partial}{\partial x_j}$ be C^∞ vector fields near the origin in (x,t,s) space that satisfy

$$M_iZ_j = \delta_{ij},\ 1 \leq i,j \leq n_0 + 1.$$

For any C^1 function $h = h(x,t,s)$,

$$dh = \sum_{i=1}^{n_0+1} M_i(h)dZ_i + \left(L_1h - \sum_{j=1}^{n_0+1} M_j(h)L_1(Z_j)\right)ds \tag{2.5}$$

which can be verified by applying both sides of the equation to the basis of vector fields $\{L_1, M_1, \cdots, M_{n_0+1}\}$ of $\mathbb{C}T(\mathbb{R}^{n_0+2})$. Equation (2.5) implies that

$$d(hdZ_1 \wedge \cdots \wedge dZ_{n_0+1}) = \left(L_1h - \sum_{j=1}^{n_0+1} M_j(h)L_1(Z_j)\right) ds \wedge dZ_1 \wedge \cdots \wedge dZ_{n_0+1}. \tag{2.6}$$

Let $q(x,t,x',t',\xi,\tau,s') = \eta(x',t')u(x',t')e^{Q(x,t,x',t',\xi,\tau,s')}$. Denoting $dZ_1\wedge\cdots\wedge dZ_{n_0+1}$ by dZ and using equation (2.6), we have,

$$d(qdZ) = \left(L_1(\eta u) + \eta u L_1(Q) - \sum_{j=1}^{n_0+1}(M_j(\eta u) + \eta u M_j(Q))L_1 Z_j\right) e^Q ds\wedge dZ. \tag{2.7}$$

By Stokes theorem, for $|s_0|$ small, we have,

$$\int_{\mathbb{R}^{n_0+1}} q(x,t,x',t',\xi,\tau,0)dx'dt' = \int_{\mathbb{R}^{n_0+1}} q(x,t,x',t',\xi,\tau,s_0)dZ(x',t',s_0) + \int_0^{s_0}\int_{\mathbb{R}^{n_0+1}} d(qdZ) \tag{2.8}$$

We will estimate the two integrals on the right in equation (2.8) for (x,t) near $(0,0)$ in $\mathbb{R}^{n_0+1}$ and (ξ,τ) in a conic neighborhood Γ of $(\xi^0,0)$ in $\mathbb{R}^{n_0+1}$. Observe that if $\psi = (\psi_1,\cdots,\psi_{n_0})$,

$$\begin{aligned}\mathrm{Re}\ Q(x,t,x',t',\xi,\tau,s') =& s'\langle\xi,\mathrm{Im}\,\psi(x',t',s')\rangle - \tau s' \\ &- K|(\xi,\tau)|(|x-x'-s'\mathrm{Re}\,\psi(x',t',s')|^2 + |t-t'|^2 - s'^2)\end{aligned} \tag{2.9}$$

Using equation (2.4), we can write

$$\begin{aligned}\mathrm{Im}\,\psi(x,t,s) =& -\frac{1}{2}\frac{\partial b}{\partial t}(x,t)s + O(s^2) \\ =& -\frac{1}{2}\frac{\partial b}{\partial t}(0,0)s + O(|xs|+|ts|+s^2)\end{aligned} \tag{2.10}$$

and so plugging this into equation (2.9) yields

$$\begin{aligned}\mathrm{Re}\,Q(x,t,x',t',\xi,\tau,s') =& -\frac{1}{2}\langle\xi,\frac{\partial b}{\partial t}(0,0)\rangle s'^2 - \tau s' \\ &- K|(\xi,\tau)|(|x-x'-s'\mathrm{Re}\,\psi(x',t',s')|^2 + |t-t'|^2 - s'^2) \\ &+ |\xi|O(|x'|s'^2 + |t'|s'^2 + |s'|^3)\end{aligned} \tag{2.11}$$

Since $\langle\xi^0,\frac{\partial b}{\partial t}(0,0)\rangle > 0$, given $0<\delta<1$, we can get $M>0$ and a conic neighborhood Γ of $(\xi^0,0)$ in $\mathbb{R}^{n_0+1}$ such that

$$\langle\xi,\frac{\partial b}{\partial t}(0,0)\rangle \geq M|\xi| \ \text{ and } \ |\tau| < \delta|\xi|, \text{ when } (\xi,\tau)\in\Gamma. \tag{2.12}$$

Our interest is in estimating the integral on the left hand side of equation (2.8) for (x,t) near $(0,0)$ and $(\xi,\tau)\in\Gamma$. When $\tau>0$, we take $s_0>0$ in (2.8) while when $\tau<0$, we use

$s_0 < 0$. This together with (2.12) allows us to deduce the following inequality from (2.11):

$$\begin{aligned} \operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \leq & -\frac{M}{2}s'^2|\xi| - K|(\xi,\tau)|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2 \\ & + |t-t'|^2 - s'^2) + |\xi|O(|x'|s'^2 + |t'|s'^2 + |s'|^3) \\ \leq & (-\frac{M}{2} + (1+\delta)K)s'^2|\xi| - K|\xi|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2 \\ & + |t-t'|^2) + |\xi|O(|x'|s'^2 + |t'|s'^2 + |s'|^3) \end{aligned} \tag{2.13}$$

Choose $K = \frac{M}{4(1+\delta)}$. Then (2.13) becomes

$$\begin{aligned} \operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \leq & -\frac{M}{4}s'^2|\xi| - \frac{M}{4(1+\delta)}|\xi|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2 \\ & + |t-t'|^2) + |\xi|O(|x'|s'^2 + |t'|s'^2 + |s'|^3). \end{aligned} \tag{2.14}$$

We choose r and $|s_0|$ small enough so that when $(x',t') \in \operatorname{supp}(\eta)$ and $|s'| \leq |s_0|, (\xi,\tau) \in \Gamma$, (2.14) will yield,

$$\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \leq -\frac{M}{8}s'^2|\xi| - \frac{M}{4(1+\delta)}|\xi|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2 + |t-t'|^2). \tag{2.15}$$

From (2.15), it follows that the first integral on the right in (2.8) (at level $s' = s_0$) decays exponentially in ξ and hence there are constants $C_1, C_2 > 0$ such that for $(\xi,\tau) \in \Gamma$,

$$\left| \int_{\mathbb{R}^{n_0+1}} q(x,t,x',t',\xi,\tau,s_0)dZ(x',t',s_0) \right| \leq C_1 e^{-C_2|(\xi,\tau)|} \tag{2.16}$$

Consider next the second integral on the right in (2.8). To estimate it, we use equation (2.7) which is a sum of two kinds of terms. The first kind consists of terms involving $L_1(Z_j), L_1(Q)$ and $L_1 u$(recall that $L_1 u = Lu = 0$) and these terms can be bounded by constant multiples of

$$|\xi||s'|^m e^{\operatorname{Re} Q(x,t,x',t',\xi,\tau,s')}, \forall m \geq 1,$$

and so using (2.15) which implies that

$$\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \leq -\frac{M}{8}s'^2|\xi|,$$

the integrals of such terms decay rapidly for $(\xi,\tau) \in \Gamma$. The second type of terms involve derivatives of $\eta(x,t)$ and hence $|x'|^2 + t'^2 \geq r^2$ in the domains of integration. Therefore, if we choose $0 < |s_0| << r$, we can get $\lambda > 0$ such that for (x,t) near $(0,0)$ and $(\xi,\tau) \in \Gamma$, (2.15) will lead to,

$$\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \leq -\lambda|\xi|, \text{ when } |x'|^2 + t'^2 \geq r^2.$$

The latter leads to an exponential decay in $(\xi,\tau) \in \Gamma$ for (x,t) near $(0,0)$ for the corresponding integrals. We conclude that there exists a neighborhood W of $(0,0)$ in (x,t) space and an open conic neighborhood Γ of $(\xi^0,0)$ in $\mathbb{R}^{n_0+1}$ such that for $\forall (x,t) \in W, (\xi,\tau) \in \Gamma, \forall m = 1,2,\cdots$, there exists $C_m > 0$ satisfying

$$\left| \int_{\mathbb{R}^{n_0+1}} e^{\sqrt{-1}[\xi(x-x')+\tau(t-t')]-K|(\xi,\tau)|(|x-x'|^2+|t-t'|^2)} \eta(x',t') u(x',t') dx'dt' \right|$$

$$= \left| \int_{\mathbb{R}^{n_0+1}} q(x,t,x',t',\xi,0) dx'dt' \right| \leq \frac{C_m}{(1+|\xi|+|\tau|)^m}.$$

By Theorem 2.1 in [BH] (see also [T] and the proof of Lemma V.5.2 in [BCH]), we conclude that

$$(\xi^0,0) \notin WF(u)|_0.$$

Suppose now the Levi form $\mathcal{L}_\sigma$ at every $\sigma \in T_p^0$ has a nonzero eigenvalue. Define

$$S = \{\sigma \in T_p^0 : \mathcal{L}_\sigma(v) \geq 0, \quad \forall v \in \mathcal{V}_p\}.$$

The set S is conic, closed and convex. If $\xi \in S$, and $\xi \neq 0$, then by hypothesis $\mathcal{L}_\xi$ has at least one positive eigenvalue and hence $-\xi \notin S$. Since $\xi \notin WF(u)$, whenever $\mathcal{L}_\xi$ has at least one negative eigenvalue, it follows that $WF(u) \subset S$, for every CR function near the point p. $\square$

Example 2.12. *Let $M = \{(z_1,z_2) \in \mathbb{C}^2 : Im\, z_2 = |z_1|^{2m}\}$ where m is a positive integer and let $M' = \{(z_1,z_2) \in \mathbb{C}^2 : Im\, z_2 = |z_1|^2\}$. Then the map $H(z_1,z_2) = (z_1^m, z_2)$ is $1-$nondegenerate at the points where $z_1 \neq 0$, and $m-$nondegenerate at all the other points. When $m > 1$, M itself is $1-$nondegenerate at the points where $z_1 \neq 0$ while when $z_1 = 0$, it is not $l-$nondegenerate for any $l \geq 0$. (The case $m = 1$ appeared in [La1]. See also [K]).*

Example 2.13. *Let $M = \{(z_1,z_2) \in \mathbb{C}^2 : Im\, z_2 = |z_1|^2\}$ and $M' = \{(w_1,w_2,w_3,w_4) \in \mathbb{C}^4 : Im\, w_4 = |w_1|^2 + |w_2|^2 - |w_3|^2\}$. For any odd positive integer $m \geq 3$, define $H_m(z_1,z_2) : M \to M$ by $H_m(z_1,z_2) = (z_1, z_2^{\frac{m}{2}}, z_2^{\frac{m}{2}}, z_2)$ where we have used a branch of the square root. H_m is a CR mapping and it is the boundary value of a holomorphic map defined on a side of M. H_m is a diffeomorphism. H_m is not smooth and so for each positive integer k, there is m such that H_m is in C^k but it is not $k-$nondegenerate.*

Example 2.14. *Let* $M = \{(z_1, z_2) \in \mathbb{C}^2 : Im\, z_2 = |z_1|^2\}$ *and* $M' = \{(w_1, w_2) \in \mathbb{C}^3 : Im\, w_3 = |w_1|^2 - |w_2|^2\}$*. For any positive integer* m*, let* $f : M \to \mathbb{C}$ *be a CR function of class* C^m *which is not smooth on any open subset of* M *(see [BX] for an example of such). Define* $H_m : M \to M'$ *by* $H_m(z_1, z_2) = (f(z_1, z_2), f(z_1, z_2), 0)$*.* H_m *is a CR mapping of class* C^m *which is not smooth on any open subset of* M*. Note that* H_m *is not* $k-$*nondegenerate for any* k*.*

3 Proof of Theorem 2.3

We begin by recalling the following "almost holomorphic" version of the implicit function theorem from [La1]:

Theorem 3.1. *Let* $U \subset \mathbb{C}^N$ *be open,* $0 \in U$*,* $A \in \mathbb{C}^p$*, and* $Z = (Z_1, \cdots, Z_N)$ *be the coordinates in* $\mathbb{C}^N$*,* W *the coordinates in* $\mathbb{C}^p$*. Let* $F : U \times \mathbb{C}^p \to \mathbb{C}^N$ *be smooth in the first* N *variables and a polynomial in the last variables. Assume that* $F(0, A) = 0$ *and* $F_Z(0, A)$ *is invertible. Then there exists a neighborhood* $U' \times V'$ *of* $(0, A)$ *and a smooth map* $\psi = (\psi_1, \cdots, \psi_N) : U' \times V' \to \mathbb{C}^N$ *with* $\psi(0, A) = 0$*, such that if* $F(Z, \overline{Z}, W) = 0$ *for some* $(Z, W) \in U' \times V'$*, then* $Z = \psi(Z, \overline{Z}, W)$*. Furthermore, for every multiindex* α*, and each* j*,* $1 \le j \le N$*,*

$$D^\alpha \frac{\partial \psi_j}{\partial Z_i}(Z, \overline{Z}, W) = 0, \quad 1 \le i \le N, \tag{3.1}$$

if $Z = \psi(Z, \overline{Z}, W)$*, and* ψ *is holomorphic in* W*. Here* D^α *denotes the derivative in all real variables.*

Given the abstract CR manifold $(M, \mathcal{V})$ of CR dimension n and CR codimension d, we will use local coordinates $(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ that vanish at $p_0 \in M$. We will write $z = (z_1, \cdots, z_n)$ where $z_j = x_j + \sqrt{-1}\, y_j$ for $j = 1, \cdots, n$. In a neighborhood W of 0, we may assume that a basis of $\mathcal{V}$ is given by $\{L_1, \cdots, L_n\}$ where

$$L_i = \frac{\partial}{\partial \overline{z}_i} + \sum_{j=1}^{n} a_{ij}(x, y, s) \frac{\partial}{\partial z_j} + \sum_{l=1}^{d} b_{il}(x, y, s) \frac{\partial}{\partial s_l}, \; 1 \le i \le n,$$

the a_{ij} and b_{il} are smooth and $a_{ij}(0) = 0 = b_{il}(0), \forall i, j, l$ (see for example [BCH], equation I.19). In these coordinates, at the origin, the characteristic set

$$T_0^0 = \{(\xi, \eta, \sigma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d : \xi = \eta = 0\}.$$

By assumption, there is an acute open convex cone $\Gamma \subset \mathbb{R}^d$ such that

$$WF(H_j)|_0 \subset \{(0, 0, \sigma) : \sigma \in \Gamma\}, \forall j = 1, \cdots, N'.$$

Let $\phi \in C_0^\infty(W)$ whose support is sufficiently small and $\phi \equiv 1$ in a neighborhood of the origin. For each $j = 1, \cdots, N'$, by Fourier's inversion formula,

$$\begin{aligned}\phi(x,y,s)H_j(x,y,s) =& \int_{\mathbb{R}^{2n+d}} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi \\ =& \int_A e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi \\ &+ \int_{\mathbb{R}^{2n+d}\setminus A} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi \\ =& I^j(x,y,s) + J^j(x,y,s)\end{aligned} \tag{3.2}$$

where $A = \{(\xi,\eta,\sigma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d : \sigma \notin \Gamma\}$.

Since $WF(H_j)|_0 \subset \{(0,0,\sigma) : \sigma \in \Gamma\}$, if the support of ϕ is sufficiently small, for every $m = 1, 2, \cdots$, there exists a constant $C_m > 0$ such that

$$|\widehat{\phi H_j}(\xi,\eta,\sigma)| \leq \frac{C_m}{(1+|\xi|+|\eta|+|\sigma|)^m}, \ \forall(\xi,\eta,\sigma) \in A.$$

It follows that $I^j(x,y,s)$ is C^∞ on $\mathbb{R}^{2n+d}$. Write

$$J^j(x,y,s) = \int_{B_1} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi + \int_{B_2} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi$$

where

$$B_1 = \{(\xi,\eta,\sigma) : |\xi|^2 + |\eta|^2 \leq 1, \ \sigma \in \overline{\Gamma}\},$$
$$B_2 = \{(\xi,\eta,\sigma) : |\xi|^2 + |\eta|^2 \geq 1, \sigma \in \overline{\Gamma}\}.$$

Observe that since $T_0^0 \cap B_2 = \emptyset$, for any CR function u near the origin, $WF(u)|_0 \cap B_2 = \emptyset$. Moreover,

$$B_2 \cap \{(\xi,\eta,\sigma) : |\xi|^2 + |\eta|^2 + |\sigma|^2 = 1\}$$

is a compact set. It follows that for each $m = 1, 2, \cdots$, we can get $C'_m > 0$ such that

$$|\widehat{\phi H_j}(\xi,\eta,\sigma)| \leq \frac{C'_m}{(1+|\xi|+|\eta|+|\sigma|)^m}, \ \forall(\xi,\eta,\sigma) \in B_2 \tag{3.3}$$

It follows that

$$F_2^j(x,y,s) = \int_{B_2} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi$$

is C^∞ on $\mathbb{R}^{2n+d}$.

Since Γ is an acute cone, there is $\sigma^0 \in \mathbb{R}^d$ such that $\sigma^0 \cdot \sigma > 0, \forall \sigma \in \Gamma$. We may assume that for some conic neighborhood Γ_1 of σ and $C_0 > 0$,

$$v \cdot \sigma \geq C_0 |v||\sigma|, \ \forall v \in \Gamma_1, \ \sigma \in \Gamma. \tag{3.4}$$

For $t \in \Gamma_1$, we define

$$F_1^j(x,y,s,t) = \int_{B_1} e^{2\pi\sqrt{-1}(x\cdot\xi + y\cdot\eta + (s+\sqrt{-1}t)\cdot\sigma)} \widehat{\phi H_j}(\xi,\eta,\sigma) d\sigma d\eta d\xi.$$

Since $\widehat{\phi H_j}$ has a polynomial growth, for some $C_1, M > 0$,

$$|\widehat{\phi H_j}(\xi,\eta,\sigma)| \leq C_1(1+|\sigma|)^M, \ \forall (\xi,\eta,\sigma) \in B_1. \tag{3.5}$$

Therefore, using (3.4) and (3.5), we get,

$$|F_1^j(x,y,s,t)| \leq C_1' \int_{\mathbb{R}^d} e^{-C_0|t||\sigma|}(1+|\sigma|)^M d\sigma \leq \frac{C_2}{|t|^{M+d+1}}, \quad t \in \Gamma_1, \text{ for some } C_1', C_2 > 0. \tag{3.6}$$

Moreover, for all multiindices $\alpha, \beta \in \mathbb{N}^n, \gamma \in \mathbb{N}^d$,

$$|\partial_x^\alpha \partial_y^\beta \partial_s^\gamma F_1^j(x,y,s,t)| \leq \frac{C}{|t|^{M+d+1+|\gamma|}}, \tag{3.7}$$

for some $C > 0$ when $t \in \Gamma_1$.

When $t \in \Gamma_1$,

$$\overline{\partial}_{w_\nu} F_1^j(x,y,s,t) = 0, \ \text{for } 1 \leq \nu \leq d, \tag{3.8}$$

where $\overline{\partial}_{w_\nu} = \frac{1}{2}(\frac{\partial}{\partial s_\nu} + \sqrt{-1}\frac{\partial}{\partial t_\nu})$.

Define

$$F_2^j(x,y,s,t) = \int_{B_2} e^{2\pi\sqrt{-1}(x\cdot\xi + y\cdot\eta + (s+\sqrt{-1}t)\cdot\sigma)} \widehat{\phi H_j}(\xi,\eta,\sigma) d\xi d\eta d\sigma,$$

for $t \in \Gamma_1$. By (3.3), F_2^j is C^∞ up to $t = 0$, and

$$\overline{\partial}_{w_\nu} F_2^j(x,y,s,t) = 0, \ \text{for } 1 \leq \nu \leq d, \ t \in \Gamma_1. \tag{3.9}$$

Since $I^j(x,y,s)$ is C^∞ and bounded, we can find a bounded C^∞ function $F_0^j(x,y,s,t)$ ($|t|$ small) such that

$$F_0^j(x,y,s,0) = I^j(x,y,s), \ \text{and } \overline{\partial}_{w_\nu} F_0^j(x,y,s,t) = O(|t|^l), \ \forall \nu = 1, \cdots, d, \forall l = 1,2,3,\ldots \tag{3.10}$$

Let $\varphi(x, y, s) \in C_0^\infty(W)$ such that its support is contained in a neighborhood of the origin where $\phi \equiv 1$. By Parseval's formula,

$$\begin{aligned}\lim_{t\to 0, t\in\Gamma_1}\int_{\mathbb{R}^{2n+d}} F_0^j(x,y,s,t)\varphi(x,y,s)dxdyds &= \int_{\mathbb{R}^{2n+d}} I^j(x,y,s)\varphi(x,y,s)dxdyds \\ &= \int_{\mathbb{R}^{2n+d}} \widehat{I^j}(\xi,\eta,\sigma)\widehat{\varphi}(-\xi,-\eta,-\sigma)d\xi d\eta d\sigma \\ &= \int_A \widehat{\phi H_j}(\xi,\eta,\sigma)\widehat{\varphi}(-\xi,-\eta,-\sigma)d\xi d\eta d\sigma\end{aligned} \tag{3.11}$$

Likewise, since F_2^j is C^∞ and bounded,

$$\int_{\mathbb{R}^{2n+d}} F_2^j(x,y,s)\varphi(x,y,s)dxdyds = \int_{B_2} \widehat{\phi H_j}(\xi,\eta,\sigma)\widehat{\varphi}(-\xi,-\eta,-\sigma)d\xi d\eta d\sigma. \tag{3.12}$$

For $t \in \Gamma_1$, using (3.4), we have,

$$\begin{aligned}&\int_{\mathbb{R}^{2n+d}} F_1^j(x,y,s,t)\varphi(x,y,s)dxdyds \\ &= \int_{B_1}\Big(\int_{\mathbb{R}^{2n+d}} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\varphi(x,y,s)dxdyds\Big)e^{-t\cdot\sigma}\widehat{\phi H_j}(\xi,\eta,\sigma)d\xi d\eta d\sigma \\ &= \int_{B_1}\widehat{\varphi}(-\xi,-\eta,-\sigma)e^{-t\cdot\sigma}\widehat{\phi H_j}(\xi,\eta,\sigma)d\xi d\eta d\sigma,\end{aligned} \tag{3.13}$$

and hence

$$\lim_{t\to 0,\ t\in\Gamma_1}\int_{\mathbb{R}^{2n+d}} F_1^j(x,y,s,t)\phi(x,y,s)dxdyds = \int_{B_1}\widehat{\varphi}(-\xi,-\eta,-\sigma)\widehat{\phi H_j}(\xi,\eta,\sigma)d\xi d\eta d\sigma \tag{3.14}$$

Let $F^j(x,y,s,t) = F_0^j(x,y,s,t) + F_1^j(x,y,s,t) + F_2^j(x,y,s,t)$ for $t \in \Gamma_1$. From (3.11),(3.12) and (3.14),

$$\begin{aligned}\lim_{t\to 0,\ t\in\Gamma_1}\int_{\mathbb{R}^{2n+d}} F^j(x,y,s,t)\varphi(x,y,s)dxdyds &= \int_{\mathbb{R}^{2n+d}}\widehat{\varphi}(-\xi,-\eta,-\sigma)\widehat{\phi H_j}(\xi,\eta,\sigma)d\xi d\eta d\sigma \\ &= \int_{\mathbb{R}^{2n+d}}\phi(x,y,s)H_j(x,y,s)\varphi(x,y,s)dxdyds.\end{aligned} \tag{3.15}$$

Therefore, in a neighborhood of the origin, in the distribution sense,

$$\lim_{t\to 0,\ t\in\Gamma_1} F^j(x,y,s,t) = H_j(x,y,s). \tag{3.16}$$

For $t \in \Gamma_1$ small, from (3.7)-(3.10), we have: for (x, y, s) near 0, given α, β, γ, there exists $C_1 > 0$ such that for some $\lambda > 0$,

$$|\partial_x^\alpha \partial_y^\beta \partial_s^\gamma F^j(x, y, s, t)| \leq \frac{C_1}{|t|^\lambda}, \text{ and} \tag{3.17}$$

$$\partial_x^\alpha \partial_y^\beta \partial_s^\gamma \overline{\partial}_{w_\nu} F^j(x, y, s, t) = O(|t|^l), \ \forall l \geq 1, \ \forall \nu = 1, \cdots, d. \tag{3.18}$$

For the rest of the proof, we follow the argument of claim 3 in [La1]. We may assume that $H(0) = 0 \in M'$. Let $\rho = (\rho_1, \cdots, \rho_{d'})$ be defining functions for M' near 0. For $\alpha \in \mathbb{N}^n$ a multiindex, recall that $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$.

Set $F(x, y, s, t) = (F^1(x, y, s, t), \cdots, F^{N'}(x, y, s, t)), t \in \Gamma_1$. As in [La1], there are smooth functions $\Psi_{\mu,\alpha}(Z', \overline{Z'}, W)$ for $|\alpha| \leq k_0, 1 \leq \mu \leq d'$, defined in a neighborhood of $\{0\} \times \mathbb{C}^{K(k_0)}$ in $\mathbb{C}^{N'} \times \mathbb{C}^{K(k_0)}$, polynomial in W, such that

$$L^\alpha \rho_\mu(H(z, s), \overline{H(z, s)}) = \Psi_{\mu,\alpha}(H(z, s), \overline{H(z, s)}, (L^\beta \overline{H}(z, s))_{|\beta| \leq k_0}), \tag{3.19}$$

and

$$L^\alpha \rho_{\mu, Z'}(H, \overline{H})|_0 = \Psi_{\mu,\alpha,Z'}(0, 0, (L^\beta \overline{H}(0, 0))_{|\beta| \leq k_0}). \tag{3.20}$$

Here $K(k_0) = N'|\{\beta : |\beta| \leq k_0\}|$. Equation (3.20) and the k_0-nondegeneracy assumption on the map H allows us to get $(\alpha^1, \cdots, \alpha^{N'}), (\mu_1, \cdots, \mu_{N'}) \in \mathbb{N}^{N'}$ and a smooth function $\psi(Z', \overline{Z'}, W) = (\psi_1, \cdots, \psi_{N'})$, which is holomorphic in W, such that with

$$\Psi = (\Psi_{\mu_1,\alpha^1}, \cdots, \Psi_{\mu_{N'},\alpha^{N'}}),$$

if $\Psi(Z', \overline{Z'}, W) = 0$, then $Z' = \psi(Z', \overline{Z'}, W)$. Moreover, with $Z' = (z'_1, \cdots, z'_{N'})$, we have,

$$D^\alpha \frac{\partial \psi_j}{\partial z'_i}(Z', \overline{Z'}, W) = 0, \ \forall i = 1, \cdots, N', \ j = 1, \cdots, N', \tag{3.21}$$

whenever $Z' = \psi(Z', \overline{Z'}, W)$. In particular, since $\Psi_{l,\alpha}(H(z, s), \overline{H}(z, s), (L^\beta \overline{H}(z, s))_{|\beta| \leq k_0}) = 0$, we have,

$$H_j(z, s) = \psi_j(F(z, s, 0), \overline{F}(z, s, 0), (L^\beta \overline{F}(z, s, 0))_{|\beta| \leq k_0}), \forall j = 1, \cdots, N'. \tag{3.22}$$

Recall that for $i = 1, \cdots, n$,

$$L_i = \frac{\partial}{\partial \overline{z}_i} + \sum_{j=1}^n a_{ij}(x, y, s) \frac{\partial}{\partial z_j} + \sum_{l=1}^d b_{il}(x, y, s) \frac{\partial}{\partial s_l}.$$

Let

$$M_i = \frac{\partial}{\partial \overline{z}_i} + \sum_{j=1}^{n} A_{ij}(x,y,s,t)\frac{\partial}{\partial z_j} + \sum_{l=1}^{d} B_{il}(x,y,s,t)\frac{\partial}{\partial s_l}, 1 \le i \le n,$$

where the A_{ij} and B_{il} are smooth extensions of the a_{ij} and b_{il} satisfying

$$\overline{\partial}_{w_\nu} A_{ij}(x,y,s,t), \overline{\partial}_{w_\nu} B_{il}(x,y,s,t) = O(|t|^m),\ \forall \nu = 1,\cdots,d,\ \forall m = 1,2,\cdots. \tag{3.23}$$

Now define

$$g_j(z,s,t) = \psi_j(F(z,s,-t), \overline{F}(z,s,-t), (M^\beta \overline{F}(z,s,-t))_{|\beta| \le k_0}),$$

for $j = 1,\cdots,N'$ and for $t \in -\Gamma_1, |t|$ small. Using (3.18),(3.21) and (3.23), we conclude that, when (z,s) is near the origin in $\mathbb{C}^n \times \mathbb{R}^d$ and $t \in -\Gamma_1$ ($|t|$ small), for any α, β, γ multiindices, there is $C > 0$ such that

$$|D_x^\alpha D_y^\beta D_s^\gamma g_j(z,s,t)| \le \frac{C}{|t|^\lambda} \text{ for some } \lambda > 0. \tag{3.24}$$

and

$$D_x^\alpha D_y^\beta D_s^\gamma \overline{\partial}_{w_\nu} g_j(z,s,t) = O(|t|^m), \forall m = 1,2,\cdots, \nu = 1,\cdots,d. \tag{3.25}$$

From (3.22), we know that,

$$H_j(z,s) = \lim_{t \to 0,\ t \in -\Gamma_1} g_j(z,s,t), \forall j = 1,\cdots,N'. \tag{3.26}$$

By Theorem V.3.7 in [BCH], it follows that $\mathrm{WF}(H_j)|_0 \cap \Gamma = \emptyset$. Since by assumption $WF(H_j)|_0 \subset \Gamma$, we conclude that H is C^∞ near the origin.

4 Proof of Theorem 2.5

Fix any $p \in M$, and assume $p' = F(p) = 0$. Since M' is strictly pseudoconvex, we may assume that there is a neighborhood G of 0 in $\mathbb{C}^{n+k}$, and a local defining function ρ of M' in G such that

$$M' \cap G = \{Z' \in G : \rho(Z', \overline{Z'}) = 0\},$$

where $\rho(Z', \overline{Z'}) = -v' + \sum_{j=1}^{n+k-1} |z'_j|^2 + \phi^*(Z', \overline{Z'})$. Here $Z' = (z'_1, \cdots, z'_{n+k})$ are the coordinates of $\mathbb{C}^{n+k}$, $z'_{n+k} = u' + \sqrt{-1}v'$ and $\phi^*(Z', \overline{Z'}) = O(|Z'|^3)$ is a real-valued smooth function on G. Note that $\mathrm{rank}_l(F,p)$ is a lower semi-continuous integer-valued function on M for each $1 \le l \le k$. For any $p \in M$,

$$\mathrm{rank}_0(F,p) \le \mathrm{rank}_1(F,p) \le \cdots \le \mathrm{rank}_k(F,p).$$

We next recall some basic properties of the rank of F. Write $F = (F_1, \cdots, F_{n+k})$. Since $F(M) \subset M'$, we have

$$\rho(F, \overline{F}) = -\frac{F_{n+k} - \overline{F_{n+k}}}{2\sqrt{-1}} + F_1\overline{F_1} + \cdots + F_{n+k-1}\overline{F_{n+k-1}} + \phi^*(F, \overline{F}) = 0, \tag{4.1}$$

on M near p. Applying $L_1, \cdots, L_n$ to the above equation, we get

$$\frac{L_j\overline{F_{n+k}}}{2\sqrt{-1}} + F_1 L_j\overline{F_1} + \cdots + F_{n+k-1}L_j\overline{F_{n+k-1}} + L_j\phi^*(F, \overline{F}) = 0,\ 1 \le j \le n, \tag{4.2}$$

$$\frac{L^\alpha\overline{F_{n+k}}}{2\sqrt{-1}} + F_1 L^\alpha\overline{F_1} + \cdots + F_{n+k-1}L^\alpha\overline{F_{n+k-1}} + L^\alpha\phi^*(F, \overline{F}) = 0, \tag{4.3}$$

on M near p for any multiindex $1 \le |\alpha| \le k$. Therefore, on M near p,

$$\rho_{Z'}(F, \overline{F}) = (\overline{F_1} + \phi^*_{z'_1}(F, \overline{F}), \cdots, \overline{F_{n+k-1}} + \phi^*_{z'_{n+k-1}}(F, \overline{F}), \frac{\sqrt{-1}}{2} + \phi^*_{z'_{n+k}}(F, \overline{F})), \tag{4.4}$$

and for any multiindex $1 \le |\alpha| \le k$,

$$L^\alpha\rho_{Z'}(F, \overline{F}) = (L^\alpha(\overline{F_1} + \phi^*_{z'_1}), \cdots, L^\alpha(\overline{F_{n+k-1}} + \phi^*_{z'_{n+k-1}}), L^\alpha\phi^*_{z'_{n+k}}). \tag{4.5}$$

Lemma 4.1. *With the assumption of Theorem 2.5, for any $p \in M$, we have* $\mathrm{rank}_0(F,p) = 1$, $\mathrm{rank}_1(F,p) = n+1$, *and thus* $\mathrm{rank}_l(F,p) \ge n+1$, *for* $1 \le l \le k$.

Proof. Assume that $F(p) = 0$. Note that $\phi^*_{z'_i}(F, \overline{F})|_p = 0$, for all $1 \le i \le n+k$. Equation (4.4) shows that $\mathrm{rank}_0(F,p) = 1$. By assumption, $dF : \mathcal{V}_p \to T_0^{(0,1)}M'$ is injective. By plugging $Z = p$ in equation (4.2), we get $L_i\overline{F}_{n+k}(p) = 0$ for each $1 \le i \le n$. Since $\{L_1, L_2, \cdots, L_n\}$ is a local basis of $\mathcal{V}$ near p, we conclude that the rank of the matrix $(L_i\overline{F}_l)_{1\le i\le n, 1\le l\le n+k-1}$ is n. Without loss of generality, we assume that

$$\begin{vmatrix} L_1\overline{F}_1 & . & . & . & L_1\overline{F}_n \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ L_n\overline{F}_1 & . & . & . & L_n\overline{F}_n \end{vmatrix} \neq 0 \text{ at } p.$$

Notice that $\phi^*_{z'_1}|_p = \phi^*_{z'_2}|_p = \cdots = \phi^*_{z'_{n+k}}|_p = 0$, $L_j\phi^*_{z'_1}|_p = L_j\phi^*_{z'_2}|_p = \cdots = L_j\phi^*_{z'_{n+k}}|_p = 0$, for all $1 \le j \le n$. Thus $\mathrm{rank}_1(F,p) = n+1$. Consequently, $\mathrm{rank}_l(F,p) \ge n+1$ for $1 \le l \le k$ for any $p \in M$. □

To simplify the notations, let

$$a_i(Z,\overline{Z}) = \overline{F}_i + \phi^*_{z'_i}(F,\overline{F}),\ 1 \le i \le n+k-1,$$

$$a_{n+k}(Z,\overline{Z}) = \frac{\sqrt{-1}}{2} + \phi^*_{z'_{n+k}}(F,\overline{F}),$$

$$\mathbf{a}(Z,\overline{Z}) = (a_1,\cdots,a_{n+k}).$$

Then

$$\rho_{Z'}(F,\overline{F}) = \mathbf{a} = (a_1,\cdots,a_{n+k-1},a_{n+k}),$$

$$L^\alpha \rho_{Z'}(F,\overline{F}) = L^\alpha \mathbf{a} = (L^\alpha a_1,\cdots,L^\alpha a_{n+k-1},L^\alpha a_{n+k})$$

for any multiindex $0 \le |\alpha| \le k$. Recall that

$$\mathrm{rank}_l(F,p) = \dim_{\mathbb{C}}(\mathrm{Span}_{\mathbb{C}}\{L^\alpha \mathbf{a}(Z,\overline{Z})|_p : 0 \le |\alpha| \le l\}).$$

The following normalization will be applied later in this section.

Lemma 4.2. *Let M, M', F be as in Theorem 2.5 and $p = 0 \in M$. Assume* $\mathrm{rank}_l(F,p) = N_0$, *for some $1 \le l \le k, n+1 \le N_0 \le n+k$. Then there exist multiindices $\{\beta_{n+1},\cdots,\beta_{N_0-1}\}$ with $1 < |\beta_i| \le l$ for all i, such that after a linear biholomorphic change of coordinates in $\mathbb{C}^{n+k}$: $\widetilde{Z} = Z'A^{-1}$, where A is a unitary $(n+k)\times(n+k)$ matrix, and $\widetilde{Z}$ denotes the new coordinates, the following hold:*

$$\widetilde{\mathbf{a}}|_p = (0,\cdots,0,\frac{\sqrt{-1}}{2}), \begin{pmatrix} L_1\widetilde{\mathbf{a}}|_p \\ \cdots \\ L_n\widetilde{\mathbf{a}}|_p \\ L^{\beta_{n+1}}\widetilde{\mathbf{a}}|_p \\ \cdots \\ L^{\beta_{N_0-1}}\widetilde{\mathbf{a}}|_p \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{N_0-1} & \mathbf{0} & \mathbf{b} \end{pmatrix}. \qquad (4.6)$$

Here we write $\widetilde{\mathbf{a}} = \widetilde{\rho}_{\widetilde{Z}}(\widetilde{Z}(F),\overline{\widetilde{Z}(F)})$, and $\widetilde{\rho}$ is a local defining function of M' near 0 in the new coordinates. Moreover, $\mathbf{B}_{N_0-1}$ is an invertible $(N_0-1)\times(N_0-1)$ matrix, $\mathbf{0}$ is an $(N_0-1)\times(n+k-N_0)$ zero matrix, and $\mathbf{b}$ is an $(N_0-1)-$dimensional column vector.

Proof. It follows from Lemma 4.1 that

$$\{\mathbf{a}, L_1\mathbf{a},\cdots,L_n\mathbf{a}\}|_p$$

is linearly independent. Extend it to a basis of $E_l(p)$, which has dimension N_0 by assumption. That is, we choose multiindices $\{\beta_{n+1}, \cdots, \beta_{N_0-1}\}$ with $1 < |\beta_i| \leq l$ for each i, such that

$$\{\mathbf{a}, L_1\mathbf{a}, \cdots, L_n\mathbf{a}, L^{\beta_{n+1}}\mathbf{a}, \cdots, L^{\beta_{N_0-1}}\mathbf{a}\}|_p$$

is linearly independent over $\mathbb{C}$. We write $\widehat{\mathbf{a}} := (a_1, \cdots, a_{n+k-1})$, that is, the first $n+k-1$ components of $\mathbf{a}$. Notice that $\mathbf{a}(p) = (0, \cdots, 0, \frac{\sqrt{-1}}{2})$. Consequently,

$$\{L_1\widehat{\mathbf{a}}, \cdots, L_n\widehat{\mathbf{a}}, L^{\beta_{n+1}}\widehat{\mathbf{a}}, \cdots, L^{\beta_{N_0-1}}\widehat{\mathbf{a}}\}|_p$$

is linearly independent in $\mathbb{C}^{n+k-1}$. Let S be the $(N_0-1)-$dimensional vector space spanned by them and let $\{T_1, \cdots, T_{N_0-1}\}$ be an orthonormal basis of S. Extend it to an orthonormal basis of $\mathbb{C}^{n+k-1}$: $\{T_1, \cdots, T_{N_0-1}, T_{N_0}, \cdots, T_{n+k-1}\}$ and set

$$T = \begin{pmatrix} T_1 \\ \cdots \\ T_{n+k-1} \end{pmatrix}^t, \qquad A = \begin{pmatrix} T & \mathbf{0}^t_{n+k-1} \\ \mathbf{0}_{n+k-1} & 1 \end{pmatrix}.$$

Here $\mathbf{0}_{n+k-1}$ is an $(n+k-1)-$dimensional zero row vector. Next we make the following change of coordinates: $Z' = \widetilde{Z}A$, or $\widetilde{Z} = Z'A^{-1}$. The function $\widetilde{\rho}(\widetilde{Z}, \overline{\widetilde{Z}}) = \rho(\widetilde{Z}A, \overline{\widetilde{Z}A})$ is a defining function of M' near 0 with respect to the new coordinates $\widetilde{Z}$. By the chain rule,

$$\widetilde{\rho}_{\widetilde{Z}}(\widetilde{Z}(F), \overline{\widetilde{Z}(F)}) = \rho_{Z'}(F, \overline{F})A. \tag{4.7}$$

For any multiindex α,

$$L^\alpha\widetilde{\rho}_{\widetilde{Z}}(\widetilde{Z}(F), \overline{\widetilde{Z}(F)}) = L^\alpha\rho_{Z'}(F, \overline{F})A. \tag{4.8}$$

In particular, at p, we get:

$$\widetilde{\mathbf{a}}|_p = \mathbf{a}|_pA, \quad \begin{pmatrix} L_1\widetilde{\mathbf{a}}|_p \\ \cdots \\ L_n\widetilde{\mathbf{a}}|_p \\ L^{\beta_{n+1}}\widetilde{\mathbf{a}}|_p \\ \cdots \\ L^{\beta_{N_0-1}}\widetilde{\mathbf{a}}|_p \end{pmatrix} = \begin{pmatrix} L_1\mathbf{a}|_p \\ \cdots \\ L_n\mathbf{a}|_p \\ L^{\beta_{n+1}}\mathbf{a}|_p \\ \cdots \\ L^{\beta_{N_0-1}}\mathbf{a}|_p \end{pmatrix} A. \tag{4.9}$$

Furthermore from the definition of A, in the new coordinates, equation (4.6) holds and $\mathbf{B}_{N_0-1}$ is invertible. □

Remark 4.3. *From the construction of A in the proof of Lemma 4.2, one can see that in the new coordinates $\widetilde{Z}$, the following continues to hold: There is a neighborhood G of $p' = 0$ in $\mathbb{C}^{n+k}$, and a smooth real-valued function $\widetilde{\rho}$ in G, such that,*

$$M' \cap G = \{\widetilde{Z} \in G : \widetilde{\rho}(\widetilde{Z}, \overline{\widetilde{Z}}) = 0\}.$$

Moreover, $\widetilde{\rho}(\widetilde{Z}, \overline{\widetilde{Z}}) = -\widetilde{v} + \sum_{j=1}^{n+k-1} |\widetilde{z}_j|^2 + \widetilde{\phi}^*(\widetilde{Z}, \overline{\widetilde{Z}})$, where $\widetilde{Z} = (\widetilde{z}_1, \cdots, \widetilde{z}_{n+k}), \widetilde{z}_{n+k} = \widetilde{u} + \sqrt{-1}\widetilde{v}$ and $\widetilde{\phi}^*(\widetilde{Z}, \overline{\widetilde{Z}}) = O(|\widetilde{Z}|^3)$ is a real-valued smooth function in G. We will write the new coordinates as Z instead of $\widetilde{Z}$.

We will next prove some lemmas on the determinants of matrices.

Lemma 4.4. *For a general $n \times n$ matrix*

$$B = \begin{pmatrix} b_{11} & b_{12} & . & . & . & b_{1n} \\ b_{21} & b_{22} & . & . & . & b_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ b_{n1} & b_{n2} & . & . & . & b_{nn} \end{pmatrix},$$

where $b_{ij} \in \mathbb{C}$ for all $1 \leq i, j \leq n, n \geq 3$, we have,

$$\begin{vmatrix} B\begin{pmatrix} 1 & 2 & . & . & . & n-2 & n-1 \\ 1 & 2 & . & . & . & n-2 & n-1 \end{pmatrix} & B\begin{pmatrix} 1 & 2 & . & . & . & n-2 & n-1 \\ j_1 & j_2 & . & . & . & j_{n-2} & n \end{pmatrix} \\ B\begin{pmatrix} i_1 & i_2 & . & . & . & i_{n-2} & n \\ 1 & 2 & . & . & . & n-2 & n-1 \end{pmatrix} & B\begin{pmatrix} i_1 & i_2 & . & . & . & i_{n-2} & n \\ j_1 & j_2 & . & . & . & j_{n-2} & n \end{pmatrix} \end{vmatrix} \qquad (*)$$

$= B\begin{pmatrix} i_1 & i_2 & . & . & . & i_{n-2} \\ j_1 & j_2 & . & . & . & j_{n-2} \end{pmatrix}|B|$, *for any $1 \leq i_1 < i_2 < \cdots < i_{n-2} \leq n-1, 1 \leq j_1 < j_2 < \cdots < j_{n-2} \leq n-1$. In particular, if $|B| = 0$, then $(*)$ equals 0. Here we have used the notation*

$$B\begin{pmatrix} i_1 & i_2 & . & . & . & i_p \\ j_1 & j_2 & . & . & . & j_p \end{pmatrix} = \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j_2} & . & . & . & b_{i_1 j_p} \\ b_{i_2 j_1} & b_{i_2 j_2} & . & . & . & b_{i_2 j_p} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ b_{i_p j_1} & b_{i_p j_2} & . & . & . & b_{i_p j_p} \end{vmatrix} \text{ for } 1 \leq p \leq n.$$

To prove Lemma 4.4, we need the following Lemmas.

Lemma 4.5. *Assume $p \geq 3$, C is a $p \times p$ matrix,*

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \cdots & \cdots & \cdots \\ c_{p1} & \cdots & c_{pp} \end{pmatrix},$$

where $c_{ij} \in \mathbb{C}$ for all $1 \leq i, j \leq p$. Then

$$c_{11}{}^{p-2}|C| = |\widetilde{C}|, \tag{4.10}$$

where $\widetilde{C}$ is a $(p-1) \times (p-1)$ matrix given by

$$\widetilde{C} = \begin{pmatrix} \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} & \cdots & \begin{vmatrix} c_{11} & c_{1p} \\ c_{21} & c_{2p} \end{vmatrix} \\ \cdots & \cdots & \cdots \\ \begin{vmatrix} c_{11} & c_{12} \\ c_{p1} & c_{p2} \end{vmatrix} & \cdots & \begin{vmatrix} c_{11} & c_{1p} \\ c_{p1} & c_{pp} \end{vmatrix} \end{pmatrix}.$$

That is, $\widetilde{C} = (\widetilde{c}_{ij})_{1 \leq i \leq (p-1), 1 \leq j \leq (p-1)}$, with $\widetilde{c}_{ij} = \begin{vmatrix} c_{11} & c_{1(j+1)} \\ c_{(i+1)1} & c_{(i+1)(j+1)} \end{vmatrix}$.

Proof. When $c_{11} = 0$, (4.10) holds since both sides equal 0. Now assume $c_{11} \neq 0$. By eliminating $c_{21}, \cdots, c_{p1}$, we get,

$$|C| = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ 0 & c_{22} - c_{12}\frac{c_{21}}{c_{11}} & \cdots & c_{2p} - c_{1p}\frac{c_{21}}{c_{11}} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & c_{p2} - c_{12}\frac{c_{p1}}{c_{11}} & \cdots & c_{pp} - c_{1p}\frac{c_{p1}}{c_{11}} \end{vmatrix} = c_{11}{}^{-(p-2)}|\widetilde{C}|. \qquad \square$$

Lemma 4.6. *If the determinant of a 3×3 matrix*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0,$$

where $a_{ij} \in \mathbb{C}$ for all $1 \leq i, j \leq 3$. Then

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} =$$

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} = 0.$$

Proof. Using Lemma 4.5,

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = a_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} = a_{12} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = a_{21} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} = a_{22} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

□

Proof of Lemma 4.4 : We proceed by induction on the dimension of B. From Lemma 4.6, we know Lemma 4.4 holds for $n = 3$. Now assume that it holds when the dimension of B is less than or equal to $n-1$. To prove it when the dimension is n, it is enough to show it for the case when $i_1 = 1, i_2 = 2, \cdots, i_{n-2} = n-2$ and $j_1 = 1, j_2 = 2, \cdots, j_{n-2} = n-2$. Namely, we show that

$$
\left|\begin{array}{cc}
\begin{vmatrix} b_{11} & b_{12} & \cdot\;\cdot\;\cdot & b_{1n-1} \\ b_{21} & b_{22} & \cdot\;\cdot\;\cdot & b_{2n-1} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ b_{n-11} & b_{n-12} & \cdot\;\cdot\;\cdot & b_{n-1n-1} \end{vmatrix} &
\begin{vmatrix} b_{11} & \cdot\;\cdot\;\cdot & b_{1n-2} & b_{1n} \\ b_{21} & \cdot\;\cdot\;\cdot & b_{2n-2} & b_{2n} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ b_{n-11} & \cdot\;\cdot\;\cdot & b_{n-1n-2} & b_{n-1n} \end{vmatrix} \\
\\
\begin{vmatrix} b_{11} & b_{12} & \cdot\;\cdot\;\cdot & b_{1n-1} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ b_{n-21} & b_{n-22} & \cdot\;\cdot\;\cdot & b_{n-2n-1} \\ b_{n1} & b_{n2} & \cdot\;\cdot\;\cdot & b_{nn-1} \end{vmatrix} &
\begin{vmatrix} b_{11} & \cdot\;\cdot\;\cdot & b_{1n-2} & b_{1n} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ b_{n-21} & \cdot\;\cdot\;\cdot & b_{n-2n-2} & b_{n-2n} \\ b_{n1} & \cdot\;\cdot\;\cdot & b_{nn-2} & b_{nn} \end{vmatrix}
\end{array}\right|
$$

$= B(\begin{smallmatrix} 1 & 2 & \cdots & n-2 \\ 1 & 2 & \cdots & n-2 \end{smallmatrix})|B|$, and the other cases are similar.

Now we view all terms here as rational functions in $b_{11}, \cdots, b_{nn}$. By Lemma 4.5,

$$
|B| = {b_{11}}^{-(n-2)} \begin{vmatrix} B(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 \\ 1 & n \end{smallmatrix}) \\ \cdots & \cdots & \cdots \\ B(\begin{smallmatrix} 1 & n \\ 1 & 2 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & n \\ 1 & n \end{smallmatrix}) \end{vmatrix} \tag{4.11}
$$

By applying Lemma 4.5 and the induction hypothesis, it follows that

$$
\begin{vmatrix} B(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 \\ 1 & n \end{smallmatrix}) \\ \cdots & \cdots & \cdots \\ B(\begin{smallmatrix} 1 & n \\ 1 & 2 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & n \\ 1 & n \end{smallmatrix}) \end{vmatrix} = \left(B(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix})\right)^{-(n-3)} {b_{11}}^{n-2} \begin{vmatrix} B(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & n \end{smallmatrix}) \\ \cdots & \cdots & \cdots \\ B(\begin{smallmatrix} 1 & 2 & n \\ 1 & 2 & 3 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 & n \\ 1 & 2 & n \end{smallmatrix}) \end{vmatrix}.
$$

Combining it with (4.11), we obtain

$$
|B| = \left(B(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix})\right)^{-(n-3)} \begin{vmatrix} B(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & n \end{smallmatrix}) \\ \cdots & \cdots & \cdots \\ B(\begin{smallmatrix} 1 & 2 & n \\ 1 & 2 & 3 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 & n \\ 1 & 2 & n \end{smallmatrix}) \end{vmatrix}.
$$

By further applications of Lemma 4.5 and the induction hypothesis as above, we arrive at the conclusion.

Finally we state the following simple lemma:

Lemma 4.7. *Let* $\mathbf{b}_1, \cdots, \mathbf{b}_n$ *and* $\mathbf{a}$ *be* n*-dimensional column vectors with entries in* $\mathbb{C}$*, and let* $B = (\mathbf{b}_1, \cdots, \mathbf{b}_n)$ *denote the* $n \times n$ *matrix. Assume that* $\det B \neq 0$*, and that* $\det(\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, \cdots, \mathbf{b}_{i_{n-1}}, \mathbf{a}) = 0$ *for any* $1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n$*. Then* $\mathbf{a} = \mathbf{0}$*, where* $\mathbf{0}$ *is the* n*-dimensional zero column vector.*

Proof. Note that $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ is a linearly independent set in $\mathbb{C}^n$. Write $\mathbf{a} = \sum_{j=1}^n \lambda_j \mathbf{b}_j$ for some $\lambda_j \in \mathbb{C}, 1 \leq j \leq n$. It is easy to see that all the $\lambda_j = 0$ by using the assumption that $\det(\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, \cdots, \mathbf{b}_{i_{n-1}}, \mathbf{a}) = 0$, $\forall 1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n$. $\square$

Theorem 2.8 will follow from:

Theorem 4.8. *Let* M, M', F *be as in Theorem 2.5 and* $p \in M$ *be a point with* $\mathrm{rank}_l(F, p) = n + l$ *for some* $1 \leq l \leq k - 1$*. Assume that in some neighborhood* O *of* p*,* $\mathrm{rank}_{l+1}(F, q) = n + l$ *for all* $q \in O$*. Then* F *is smooth near* p*.*

Proof. Assume $p = 0$. Applying Lemma 4.2, after a suitable holomorphic change of coordinates in $\mathbb{C}^{n+k}$, there exist multiindices $\{\beta_{n+1}, \cdots, \beta_{n+l-1}\}$ with $1 < |\beta_i| \leq l$ for all $n \leq i \leq n + l - 1$ satisfying

$$\mathbf{a}|_p = (0, \cdots, 0, \frac{\sqrt{-1}}{2}), \begin{pmatrix} L_1 \mathbf{a}|_p \\ \cdots \\ L_n \mathbf{a}|_p \\ L^{\beta_{n+1}} \mathbf{a}|_p \\ \cdots \\ L^{\beta_{n+l-1}} \mathbf{a}|_p \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{n+l-1} & \mathbf{0} & \mathbf{b} \end{pmatrix}. \tag{4.12}$$

Here $\mathbf{B}_{n+l-1}$ is an invertible $(n + l - 1) \times (n + l - 1)$ matrix, $\mathbf{0}$ is an $(n + l - 1) \times (k - l)$ zero matrix, $\mathbf{b}$ is an $(n + l - 1)-$dimensional column vector. From equation (4.12), we know that

$$\begin{vmatrix} a_1 & \cdots & a_{n+l-1} & a_{n+k} \\ L_1 a_1 & \cdots & L_1 a_{n+l-1} & L_1 a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L_n a_1 & \cdots & L_n a_{n+l-1} & L_n a_{n+k} \\ L^{\beta_{n+1}} a_1 & \cdots & L^{\beta_{n+1}} a_{n+l-1} & L^{\beta_{n+1}} a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_1 & \cdots & L^{\beta_{n+l-1}} a_{n+l-1} & L^{\beta_{n+l-1}} a_{n+k} \end{vmatrix} \neq 0 \text{ at } p. \tag{4.13}$$

To simplify the notation, we denote the $n-$dimensional multiindices by $\beta_0 = (0, \cdots, 0)$, and $\beta_\mu = (0, \cdots, 0, 1, 0, \cdots, 0)$, for $\mu = 1, \cdots, n$, where 1 is at the μ^{th} position. That is, $L^{\beta_\mu} = L_\mu, \mu = 1, \cdots, n$. Then inequality (4.13) can be written as

$$\begin{vmatrix} L^{\beta_0} a_1 & \cdots & L^{\beta_0} a_{n+l-1} & L^{\beta_0} a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_1 & \cdots & L^{\beta_{n+l-1}} a_{n+l-1} & L^{\beta_{n+l-1}} a_{n+k} \end{vmatrix} \neq 0 \text{ at } p. \tag{4.14}$$

By shrinking O if necessary, it is nonzero everywhere in O. Since $\text{rank}_{l+1}(F, q) \leq n + l$ in O, we have

$$\dim_{\mathbb{C}}(E_{l+1}(q)) = \dim_{\mathbb{C}}(\text{Span}_{\mathbb{C}}\{(L^\alpha a_1, \cdots, L^\alpha a_{n+k})|_q : 0 \leq |\alpha| \leq l + 1\}) \leq n + l$$

everywhere in O. Hence for any multiindex $\widetilde{\beta}$ with $0 \leq |\widetilde{\beta}| \leq l+1$, and any $n+l \leq j \leq n+k-1$, we have, in O,

$$\begin{vmatrix} L^{\beta_0} a_1 & \cdots & L^{\beta_0} a_{n+l-1} & L^{\beta_0} a_{n+k} & L^{\beta_0} a_j \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_1 & \cdots & L^{\beta_{n+l-1}} a_{n+l-1} & L^{\beta_{n+l-1}} a_{n+k} & L^{\beta_{n+l-1}} a_j \\ L^{\widetilde{\beta}} a_1 & \cdots & L^{\widetilde{\beta}} a_{n+l-1} & L^{\widetilde{\beta}} a_{n+k} & L^{\widetilde{\beta}} a_j \end{vmatrix} \equiv 0. \tag{4.15}$$

Furthermore, we will prove the following claim.

Claim: For any $1 \leq \nu \leq n, n + l \leq j \leq n + k - 1$, and $i_1 < i_2 < \cdots < i_{n+l-1}$ with $\{i_1, \cdots, i_{n+l-1}\} \subset \{1, \cdots, n + l - 1, n + k\}$, the following holds in O :

$$L_\nu \left(\frac{\begin{vmatrix} L^{\beta_0} a_{i_1} & \cdots & L^{\beta_0} a_{i_{n+l-1}} & L^{\beta_0} a_j \\ L^{\beta_1} a_{i_1} & \cdots & L^{\beta_1} a_{i_{n+l-1}} & L^{\beta_1} a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_{i_1} & \cdots & L^{\beta_{n+l-1}} a_{i_{n+l-1}} & L^{\beta_{n+l-1}} a_j \end{vmatrix}}{\begin{vmatrix} L^{\beta_0} a_1 & \cdots & L^{\beta_0} a_{n+l-1} & L^{\beta_0} a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_1 & \cdots & L^{\beta_{n+l-1}} a_{n+l-1} & L^{\beta_{n+l-1}} a_{n+k} \end{vmatrix}} \right) \equiv 0. \tag{4.16}$$

Proof. By the quotient rule,

$$\text{the numerator of } \left(L_\nu \left(\frac{\begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix}}{\begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix}} \right) \right) =$$

.

$$\begin{vmatrix} \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \\ L_\nu \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & L_\nu \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \end{vmatrix} =$$

$$\begin{vmatrix} \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \\ \begin{vmatrix} L_\nu L^{\beta_0}a_1 & \cdots & L_\nu L^{\beta_0}a_{n+l-1} & L_\nu L^{\beta_0}a_{n+k} \\ L^{\beta_1}a_1 & \cdots & L^{\beta_1}a_{n+l-1} & L^{\beta_1}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L_\nu L^{\beta_0}a_{i_1} & \cdots & L_\nu L^{\beta_0}a_{i_{n+l-1}} & L_\nu L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \end{vmatrix} + \cdots +$$

$$\begin{vmatrix} \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \\ \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-2}}a_1 & \cdots & L^{\beta_{n+l-2}}a_{n+l-1} & L^{\beta_{n+l-2}}a_{n+k} \\ L_\nu L^{\beta_{n+l-1}}a_1 & \cdots & L_\nu L^{\beta_{n+l-1}}a_{n+l-1} & L_\nu L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-2}}a_{i_1} & \cdots & L^{\beta_{n+l-2}}a_{i_{n+l-1}} & L^{\beta_{n+l-2}}a_j \\ L_\nu L^{\beta_{n+l-1}}a_{i_1} & \cdots & L_\nu L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L_\nu L^{\beta_{n+l-1}}a_j \end{vmatrix} \end{vmatrix}.$$

From equation (4.15) and Lemma 4.4, we know each term on the right-hand side of the equation above equals 0. Hence equation (4.16) holds. This completes the proof of the claim. □

Thus the fraction in the parentheses in equation (4.16) equals a C^{k-l} CR function in O. It follows that for any fixed $n+l \le j \le n+k-1$, there exist $C^{k-l}-$smooth CR functions $G_1^j, G_2^j, \cdots, G_{n+l-1}^j, G_{n+k}^j$ in O, such that, if $i_1 < i_2 < \cdots < i_{n+l-1}$ and $(i_1, i_2, \cdots, i_{n+l-1}) = (1, 2, \cdots, \widehat{i_0}, \cdots, n+l-1, n+k), i_0 \in \{1, 2, \cdots, n+l-1, n+k\}$ (where $(1, 2, \cdots, \widehat{i_0}, \cdots, n+l-1, n+k)$ means $(1, 2, \cdots, n+l-1, n+k)$ with the component "i_0" missing) then in O,

$$\begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix}$$

$$= G_{i_0}^j \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_{i_0} \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_{i_0} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_{i_0} \end{vmatrix}.$$

That is,

$$\begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}(a_j - G_{i_0}^j a_{i_0}) \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}(a_j - G_{i_0}^j a_{i_0}) \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}(a_j - G_{i_0}^j a_{i_0}) \end{vmatrix} \equiv 0. \tag{4.17}$$

We further assert:

Claim: In O, we have,

$$\begin{vmatrix} L^{\beta_0}a_{s_1} & \cdots & L^{\beta_0}a_{s_{n+l-1}} & L^{\beta_0}(a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k}) \\ L^{\beta_1}a_{s_1} & \cdots & L^{\beta_1}a_{s_{n+l-1}} & L^{\beta_1}(a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k}) \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{s_1} & \cdots & L^{\beta_{n+l-1}}a_{s_{n+l-1}} & L^{\beta_{n+l-1}}(a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k}) \end{vmatrix} \equiv 0 \tag{4.18}$$

for all $s_1 < s_2 < \cdots < s_{n+l-1}$ with $\{s_1, \cdots, s_{n+l-1}\} \subset \{1, \cdots, n+l-1, n+k\}$ and any $n+l \le j \le n+k-1$.

Proof. Assume that $(s_1, \cdots, s_{n+l-1}) = (1, \cdots, \widehat{s_0}, \cdots, n+l-1, n+k)$. Notice that for any $n+l \le j \le n+k-1, i \neq s_0$ and $i \in \{1, \cdots, n+l-2, n+k\}$,

$$\begin{vmatrix} L^{\beta_0} a_{s_1} & \cdots & L^{\beta_0} a_{s_{n+l-1}} & L^{\beta_0}(G_i^j a_i) \\ L^{\beta_1} a_{s_1} & \cdots & L^{\beta_1} a_{s_{n+l-1}} & L^{\beta_1}(G_i^j a_i) \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_{s_1} & \cdots & L^{\beta_{n+l-1}} a_{s_{n+l-1}} & L^{\beta_{n+l-1}}(G_i^j a_i) \end{vmatrix} \equiv 0. \tag{4.19}$$

Combining this with equation (4.17), one can check that equation (4.18) holds. □

By Lemma 4.7, equation (4.14), and (4.18), we immediately obtain that in O,

$$L^{\beta_t}(a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k}) = 0, \forall\ 1 \le t \le n+l-1, n+l \le j \le n+k-1.$$

In particular, when $t = 0$, we have:

$$a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k} = 0, n+l \le j \le n+k-1. \tag{4.20}$$

That is, in O,

$$F_j + \overline{\phi^*_{z'_j}} - \sum_{i=1}^{n+l-1} \overline{G_i^j}(F_i + \overline{\phi^*_{z'_i}}) - \overline{G_{n+k}^j}(\frac{1}{2\sqrt{-1}} + \overline{\phi^*_{z'_{n+k}}}) = 0. \tag{4.21}$$

Recall that we have, by shrinking O if necessary, in O,

$$-\frac{F_{n+k} - \overline{F_{n+k}}}{2\sqrt{-1}} + F_1\overline{F_1} + \cdots + F_{n+k-1}\overline{F_{n+k-1}} + \phi^*(F, \overline{F}) = 0, \tag{4.22}$$

$$\frac{L_j \overline{F_{n+k}}}{2\sqrt{-1}} + F_1 L_j \overline{F_1} + \cdots + F_{n+k-1} L_j \overline{F_{n+k-1}} + L_j \phi^*(F, \overline{F}) = 0, 1 \le j \le n, \tag{4.23}$$

$$\frac{L^{\beta_t} \overline{F_{n+k}}}{2\sqrt{-1}} + F_1 L^{\beta_t} \overline{F_1} + \cdots + F_{n+k-1} L^{\beta_t} \overline{F_{n+k-1}} + L^{\beta_t} \phi^*(F, \overline{F}) = 0, n+1 \le t \le n+l-1. \tag{4.24}$$

We introduce local coordinates $(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ that vanish at the central ponit $p \in M$. By Theorem 2.9, $G_i^j, G_{n+k}^j, F_1, \cdots, F_{n+k}$ extend to almost analytic functions into a wedge $\{(x, y, s+it) \in U \times V \times \Gamma_1 : (x, y, s) \in U \times V,\ t \in \Gamma_1\}$, with edge M near $p = 0$ for all $1 \le i \le n+l-1, n+l \le j \le n+k-1$. Here $U \times V$ is a neighborhood of the origin in $\mathbb{C}^n \times \mathbb{R}^d$

and Γ_1 is an acute convex cone in $\mathbb{R}^d$ in $t-$space. We still denote the extended functions by $G_i^j, G_{n+k}^j, F_1, \cdots, F_{n+k}$. Arguments similar to those used in the proof of Theorem 2.3 imply that the G_i^j and G_{n+k}^j satisfy the estimates:

$$\left|D_x^\alpha D_y^\beta D_s^\gamma G_i^j(z,s,t)\right| \leq \frac{C}{|t|^\lambda}, \left|D_x^\alpha D_y^\beta D_s^\gamma G_{n+k}^j(z,s,t)\right| \leq \frac{C}{|t|^\lambda}, \text{ for some } C, \lambda > 0$$

and

$$D_x^\alpha D_y^\beta D_s^\gamma \overline{\partial}_{w_\nu} G_i^j(z,s,t) = O(|t|^m), D_x^\alpha D_y^\beta D_s^\gamma \overline{\partial}_{w_\nu} G_{n+k}^j(z,s,t) = O(|t|^m),$$

for all $1 \leq i \leq n+l-1, n+l \leq j \leq n+k-1, 1 \leq \nu \leq d, m \geq 1$. Similar estimates hold for $F_1, \cdots, F_{n+k}$.

We now use equations (4.22), (4.23), (4.24) and (4.21) to get a smooth map $\Psi(Z', \overline{Z'}, W) = (\Psi_1, \cdots, \Psi_{n+k})$ defined in a neighborhood of $\{0\} \times \mathbb{C}^q$ in $\mathbb{C}^{n+k} \times \mathbb{C}^q$, smooth in the first $n+k$ variables and polynomial in the last q variables for some integer q, such that,

$$\Psi(F, \overline{F}, (L^\alpha \overline{F})_{1 \leq |\alpha| \leq l}, \overline{G_1^{n+l}}, \cdots, \overline{G_{n+l-1}^{n+l}}, \overline{G_{n+k}^{n+l}}, \cdots, \overline{G_1^{n+k-1}}, \cdots, \overline{G_{n+l-1}^{n+k-1}}, \overline{G_{n+k}^{n+k-1}}) = 0$$

at $(z,s,0)$ with $(z,s) \in U \times V$. Write

$$\overline{G} = (\overline{G_1^{n+l}}, \cdots, \overline{G_{n+l-1}^{n+l}}, \overline{G_{n+k}^{n+l}}, \cdots, \overline{G_1^{n+k-1}}, \cdots, \overline{G_{n+l-1}^{n+k-1}}, \overline{G_{n+k}^{n+k-1}}).$$

Observe that

$$\Psi_{Z'}|_{(p,(L^\alpha \overline{F})_{1 \leq |\alpha| \leq l}(p), \overline{G}(p))} = \begin{pmatrix} \mathbf{0}_{n+l-1} & \mathbf{0}_{k-l} & \frac{\sqrt{-1}}{2} \\ \mathbf{B}_{n+l-1} & \mathbf{0} & \mathbf{b} \\ \mathbf{C} & \mathbf{I}_{k-l} & \mathbf{0}_{k-l}^t \end{pmatrix},$$

where $\mathbf{0}_N$ is an $N-$dimensional zero row vector, $\mathbf{C}$ is a $(k-l) \times (n+l-1)$ matrix, $\mathbf{I}_{k-l}$ is the $(k-l) \times (k-l)$ identity matrix and we recall that $\mathbf{B}_{n+l-1}$ is an invertible $(n+l-1) \times (n+l-1)$ matrix, $\mathbf{0}$ is an $(n+l-1) \times (k-l)$ zero matrix, $\mathbf{b}$ is an $(n+l-1)-$dimensional column vector.

The matrix $\Psi_{Z'}|_{(p,(L^\alpha \overline{F})_{1 \leq |\alpha| \leq l}(p), \overline{G}(p))}$ is invertible. By applying Theorem 3.1, we get a solution $\psi = (\psi_1, \cdots, \psi_{n+k})$ satisfying (3.1) and for each $1 \leq j \leq n+k$,

$$F_j = \psi_j(F, \overline{F}, (L^\alpha \overline{F})_{1 \leq |\alpha| \leq l}, \overline{G})$$

at $(z,s,0)$ with $(z,s) \in U \times V$. Recall that in the proof of Theorem 2.3, for each $i = 1, \cdots, n$, we denoted by M_i the smooth extension of L_i to $U \times V \times \mathbb{R}^d$ satisfying (3.23). For each $1 \leq j \leq n+k$, set

$$h_j(z,s,t) = \psi_j(F(z,s,-t), \overline{F}(z,s,-t), (M^\alpha \overline{F})_{1 \leq |\alpha| \leq l}(z,s,-t), \overline{G}(z,s,-t))$$

and shrink U and V and choose δ in such a way that h_j is well defined and continuous in $\overline{\Omega_-}$ where $\Omega_- = \{(x,y,s+it) : (x,y,s) \in U \times V,\, t \in -\Gamma_1, |t| \leq \delta\}$. The same proof as before leads to the estimates:

$$\left|D_x^\alpha D_y^\beta D_s^\gamma h_j(z,s,t)\right| \leq \frac{C}{|t|^\lambda}, \text{ for some } C, \lambda > 0$$

and

$$D_x^\alpha D_y^\beta D_s^\gamma \overline{\partial}_{w_\nu} h_j(z,s,t) = O(|t|^m),\ \forall \nu = 1, \cdots, d,\ m = 1, 2, \ldots$$

for $t \in -\Gamma_1$, $1 \leq j \leq n+k$.

Notice that the F_j satisfy similar estimates in Γ_1, and $b_+ F_j = b_- h_j$ for each $1 \leq j \leq n+k$. Applying Theorem V.3.7 in [BCH] as before, we conclude that F is smooth near p. This establishes Theorem 4.8.

□

Proof of Theorem 2.8: Fix $p \in \Omega_2$. Let a neighborhood $\widetilde{O}$ of p and $\{p_i\}_0^\infty \subset \widetilde{O}$ be as mentioned in Remark 2.7, and write $d = \deg(F,p)$. Since $\mathrm{rank}_d(F,q) \leq n+d-1$ for all $q \in \widetilde{O}$, and $\mathrm{rank}_{d-1}(F,p_i) = n+d-1$ for all $i \geq 0$, by Theorem 4.8, F is smooth near p_i for all $i \geq 0$. This establishes Theorem 2.8.

Theorem 2.5 follows from Theorem 2.8 and Theorem 2.9.

As a consequence of Theorem 4.8, we immediately have

Corollary 4.9. *Let $M \subset \mathbb{C}^{n+1}$, $M' \subset \mathbb{C}^{n+k}$ be two smooth strongly pseudoconvex real hypersurfaces $(n \geq 1, k \geq 1)$, $F : M \to M'$ be a C^2-smooth CR map. Assume that* $\mathrm{rank}_2(F,p) \leq n+1$ *everywhere in M. Then F is smooth.*

Proof. We may assume that F is nonconstant. By a well known argument using Hopf's lemma as in the Appendix, $dF : T_p^{(1,0)}M \to T_{F(p)}^{(1,0)}M'$ is injective at every $p \in M$. Note that $\mathrm{rank}_1(F,p) = n+1$ for all $p \in M$ by Lemma 4.1. By Theorem 4.8 (note that in this case, the proof showed that we did not need F to be C^k), we arrive at the conclusion. □

Since a CR diffeomorphism of class C^k of a $k-$nondegenerate manifold is $k-$nondegenerate, Thoerem 2.3 implies the following:

Corollary 4.10. *Let $M \subset \mathbb{C}^N$ be a generic CR manifold that is k_0-nondegenerate. Suppose $H = (H_1, \cdots, H_N) : M \to M$ is a CR diffeomorphism of class C^{k_0} such that for some $p_0 \in M$ and an open convex cone $\Gamma \subset \mathbb{R}^d$,*

$$\mathrm{WF}(H_j)|_{p_0} \subset \Gamma, j = 1, \cdots, N$$

where d is the CR codimension of M. Then H is C^∞ in some neighborhood of p_0.

5 Appendix

5.1 On CR mappings into a lower dimensional target

In this appendix we include a result which shows why we don't consider the case when the target manifold has a lower CR dimension. The result is known to experts but we have presented it here since we are not aware of a reference.

Theorem 5.1. *Let $M \subset \mathbb{C}^N, M' \subset \mathbb{C}^{N'}$ be smooth strongly pseudoconvex real hypersurfaces with $N \geq 2, N' \geq 2$. Let $F : M \to M'$ be a CR mapping of class C^2. Assume that $N' < N$. Then F is a constant map.*

Proof. Suppose that F is nonconstant. Fix $p \in M, p' \in M'$ with $p' = F(p)$. Choose suitable coordinates in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ near p, p' such that: $p = 0, p' = 0$ and M is locally defined by

$$r(Z, \overline{Z}) = -\frac{z_N - \overline{z_N}}{2\sqrt{-1}} + z_1\overline{z}_1 + \cdots + z_{N-1}\overline{z_{N-1}} + \phi(Z, \overline{Z}) \text{ near } p,$$

M' is locally defined by

$$\rho(Z', \overline{Z'}) = -\frac{z'_{N'} - \overline{z'_{N'}}}{2\sqrt{-1}} + z'_1\overline{z'}_1 + \cdots + z'_{N'-1}\overline{z'_{N'-1}} + \phi^*(Z', \overline{Z'}) \text{ near } p'.$$

Here $Z = (z_1, \cdots, z_N), Z' = (z'_1, \cdots, z'_{N'})$ are the coordinates of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively. Moreover, $\phi(Z, \overline{Z}) = O(|Z|^3), \phi^*(Z', \overline{Z'}) = O(|Z'|^3)$ are real-valued smooth functions near p, p' respectively. We write

$$L_i = \frac{\partial r}{\partial z_N}\frac{\partial}{\partial z_i} - \frac{\partial r}{\partial z_i}\frac{\partial}{\partial z_N}, \quad T = \sqrt{-1}(\frac{\partial r}{\partial \overline{z}_N}\frac{\partial}{\partial z_N} - \frac{\partial r}{\partial z_N}\frac{\partial}{\partial \overline{z}_N}), 1 \leq i \leq N-1,$$

which are vector fields tangent to M near the central point 0.

Write $F = (F_1, \cdots, F_{N'})$. Near 0 on M we have

$$\rho(F, \overline{F}) = -\frac{F_{N'} - \overline{F_{N'}}}{2\sqrt{-1}} + F_1\overline{F}_1 + \cdots + F_{N'-1}\overline{F_{N'-1}} + \phi^*(F, \overline{F}) = 0. \tag{5.1}$$

Applying $L_i, 1 \leq i \leq N-1$ to equation (5.1) and evaluating at 0, one easily gets

$$\frac{\partial F_{N'}}{\partial z_i}|_0 = 0, 1 \leq i \leq N-1.$$

Similarly, by applying $L_k L_l, 1 \leq k, l \leq N-1$ to equation (5.1) and evaluating at 0, we get,

$$\frac{\partial^2 F_{N'}}{\partial z_k \partial z_l}|_0 = 0, 1 \leq k, l \leq N-1.$$

By the Lewy extension theorem, F extends holomorphically to the pseudoconvex side of M denoted by Ω. We may assume that Ω is a union of analytic discs attached to M. That is, for each $q \in \Omega$, there exists a continuous function

$$G : \overline{\Delta} \to \mathbb{C}^N \quad (\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\})$$

analytic on Δ such that $G(\Delta) \subset \Omega$, $G(\partial\Delta) \subset M$, and $G(0) = q$. For each such G, the function $\rho \circ F \circ G$ is continuous on $\overline{\Delta}$, subharmonic on Δ and vanishes on $\partial\Delta$. If this function is constant for every analytic disc G attached to M, then F would map Ω into M' which would contradict the strict pseudoconvexity of M' unless F is constant. This allows us to apply the maximum principle and the Hopf lemma to the subharmonic function $\rho(F, \overline{F}) \leq 0$ near p over Ω to conclude that

$$\frac{\partial}{\partial \mathrm{Im}(z_N)}(\rho(F, \overline{F}))|_0 = \frac{\partial}{\partial \mathrm{Im}(z_N)}(-\mathrm{Im}(F_{N'}) + \sum_{j=1}^{N'-1} |F_j|^2)|_0 = -\lambda < 0, \tag{5.2}$$

for some $\lambda > 0$, that is,

$$\sqrt{-1}(\frac{\partial}{\partial z_N} - \frac{\partial}{\partial \overline{z_N}})(-\frac{F_{N'} - \overline{F}_{N'}}{2\sqrt{-1}})|_0 = -\frac{1}{2}(\frac{\partial F_{N'}}{\partial z_N} + \frac{\partial \overline{F}_{N'}}{\partial \overline{z}_N})|_0 = -\frac{\partial F_{N'}}{\partial z_N}|_0 = -\lambda < 0. \tag{5.3}$$

Here we have used the fact that

$$T(\rho(F, \overline{F})) = 0 \tag{5.4}$$

which implies $\frac{\partial F_{N'}}{\partial z_N}|_0 = \frac{\partial \overline{F}_{N'}}{\partial \overline{z}_N}|_0$. Hence we can write:

$$F_{N'} = \lambda z_N + O(|z_N||\tilde{z}| + |z_N|^2) + o(|Z|^2), \;\; \tilde{z} = (z_1, \ldots, z_{N-1}) \tag{5.5}$$

$$F_j = b_j z_N + \sum_{i=1}^{N-1} a_{ij} z_i + O(|Z|^2), 1 \leq j \leq N'-1, \tag{5.6}$$

where $b_j \in \mathbb{C}, a_{ij} \in \mathbb{C}, 1 \leq i \leq N-1, 1 \leq j \leq N'-1$. That is,

$$(F_1, \cdots, F_{N'-1}) = z_N(b_1, \cdots, b_{N'-1}) + (z_1, \cdots, z_{N-1})A + (\hat{F}_1, \cdots, \hat{F}_{N'-1}), \tag{5.7}$$

where $A = (a_{ij})_{(N-1)\times(N'-1)}$ is an $(N-1)\times(N'-1)$ matrix, and $\hat{F}_j = O(|Z|^2)$, for any $1 \le j \le N'-1$. Next we write $Z = (\widetilde{z}, z_N)$, where $\widetilde{z} = (z_1, \cdots, z_{N-1})$, and we introduce the notion of weighted degree: For a function h on M, we write $h \in o_{wt}(s)$ if

$$\lim_{t\to 0^+} \frac{h(t\widetilde{z}, t^2 z_N, t\overline{\widetilde{z}}, t^2\overline{z}_N)}{t^s} \to 0$$

uniformly with respect to $(\widetilde{z}, z_N) \approx 0$ in $\mathbb{C}^{N-1}\times\mathbb{C}$. That is, we equip $\widetilde{z}, z_N$ with weighted degrees $1, 2$ respectively.

From equation (5.1)

$$\frac{F_{N'} - \overline{F}_{N'}}{2\sqrt{-1}} = (F_1, \cdots, F_{N'-1})(\overline{F}_1, \cdots, \overline{F}_{N'-1})^t + \phi^*(F, \overline{F}), \tag{5.8}$$

whenever $z_N = u + \sqrt{-1}(|\widetilde{z}|^2 + \phi(Z, \overline{Z}))$ near 0. We can rewrite equation (5.8) in terms of u and $\widetilde{z}$ by using equations (5.5),(5.6),(5.7):

$$\lambda|\widetilde{z}|^2 + o_{wt}(2) = (z_1, \cdots, z_{N-1})AA^*(\overline{z}_1, \cdots, \overline{z}_{N-1})^t + o_{wt}(2) \tag{5.9}$$

Then by collecting terms on both sides of weighted degree two, one easily gets,

$$\lambda|\widetilde{z}|^2 = (z_1, \cdots, z_{N-1})AA^*(\overline{z}_1, \cdots, \overline{z}_{N-1})^t,$$

which implies that,

$$\lambda \mathbf{I}_{N-1} = AA^*, \tag{5.10}$$

where $\mathbf{I}_{N-1}$ is the $(N-1)\times(N-1)$ identity matrix. But A is an $(N-1)\times(N'-1)$ matrix with rank at most $N'-1$ and so (5.10) can not hold since $N'-1 < N-1$.

□

References

[BER] S. Baouendi, P. Ebenfelt, and L. Rothschild, Real submanifolds in complex space and their mappings, *Princeton University Press,* Princeton, (1999).

[BHR] M. S. Baouendi, X. Huang and L. Rothschild, Regularity of CR mappings between algebraic hypersurfaces, *Invent. Math.*, 125, 13-36 (1996).

[BJT] M. S. Baouendi, H. Jacobowitz and F. Treves, On the analyticity of CR mappings, *Annals of Mathematics*, 122, 365-400 (1985).

[BR] M. S. Baouendi and L. Rothschild, Remarks on the generic rank of a CR mapping, *J. Geom. Anal.*, 2, 1-9 (1992).

[Be] E. Bedford, Proper holomorphic mappings, *Bull. Amer. Math. Soc.*, 10, 157-175 (1984).

[BN] S. Bell and R. Narasimhan, Proper holomorphic mappings of complex spaces, *EMS 69, Several Complex Variables VI*, Springer-Verlag, (1990).

[BCH] S. Berhanu, P. Cordaro, and J. Hounie, An introduction to involutive structure, *Cambridge University Press*, (2008)

[BH] S. Berhanu and J. Hounie, An F. and M. Riesz theorem for planar vector fields, *Math. Ann.*, 320, 463-485 (2001).

[BX] S. Berhanu and M. Xiao, On the regularity of CR mappings between CR manifolds of hypersurface type, preprint.

[CKS] J. A. Cima, S. Krantz, and T. J. Suffridge, A reflection principle for proper holomorphic mappings of strongly pseudoconvex domains and applications, *Math Z.*, 186, 1-8 (1984).

[CS] J. A. Cima and T. J. Suffridge, A reflection principle with applications to proper holomorphic mappings, *Math Ann.*, 265, 489-500 (1983).

[CGS] B. Coupet, H. Gaussier, and A. Sukhov, Regularity of CR maps between convex hypersurfaces of finite type, *Proc. Amer. Math. Soc.*, 127, 3191-3200 (1999).

[DW] K. Diedrich and S. Webster, A reflection principle for degenerate hypersurfaces, *Duke Math J.*, 47, 835-843 (1980).

[E] P. Ebenfelt, New invariant tensors in CR structures and a normal form for real hypersurfaces at a generic Levi degeneracy, *J. Differential Geometry* , 50, 207-247 (1998).

[EH] P. Ebenfelt and X. Huang, On a generalized reflection principle in $\mathbb{C}^2$, *Ohio State Univ. Math. Res. Inst. Publ.*, 9, 125-139 (2001).

[EL] P. Ebenfelt and B. Lamel, Finite jet determination of CR embeddings, *J. Geom. Anal.*, 14, 241-265 (2004).

[Fe] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Invent. Math.*, 26, 1-65 (1974).

[Fr1] F. Forstneric, Extending proper holomorphic mappings of positive codimension, *Invent. Math.*, 95, 31-62 (1989).

[Fr2] F. Forstneric, A survey on proper holomorphic mappings, Proceeding of Year in SCVs at Mittag-Leffler Institute, *Math. Notes* 38, Princeton, NJ: Princeton University Press, (1992).

[Fr3] F. Forstneric, Mappings of strongly pseudoconvex Cauchy-Riemann manifolds, Several complex variables and complex geometry, *Proc. Sympos. Pure Math 52, Part 1*, Amer. Math. Soc., Providence, RI, 59-92 (1991).

[H] L. Hormander, The Analysis of Linear Partial Differential Operators: I. Distribution theory and Fourier analysis., *Springer-Verlag*, Berlin, (1990)

[Hu1] X. Huang, On the mapping problem for algebraic real hypersurfaces in the complex spaces of different dimensions, *Ann. Inst. Fourier*, 44, 433-463 (1994).

[Hu2] X. Huang, Geometric analysis in several complex variables, Ph.D. Thesis, Washington University in St. Louis (1994).

[J] H. Jacobowitz, An introduction to CR structures, *American Mathematical Society*, (1990)

[KP] S.V. Khasanov and S. I. Pinchuk, Asymptotically holomorphic functions and their applications, *Mat. Sb (N.S.)*, 134(176), 546-555 (1987).

[K] S. Kim, Complete system of finite order for CR mappings between real analytic hyper surfaces of degenerate Levi form, *J. Korean Math. Soc.*, 38, 87-99 (2001).

[La1] B. Lamel, A $C^\infty-$regularity for nondegenerate CR mappings, *Monatsh. Math.*, 142, 315-326 (2004).

[La2] B. Lamel, A reflection principle for real-analytic submanifolds of complex spaces, *J. Geom. Anal.*, 11, 625-631 (2001).

[La3] B. Lamel, Holomorphic maps of real submanifolds in complex spaces of different dimensions, *Pac. J. Math.*, 201, 357-387 (2001).

[Le] H. Lewy, On the boundary behavior of holomorphic mappings, *Acad. Naz. Lincei*, 3, 1-8 (1977).

[M] N. Mir, An algebraic characterization of holomorphic nondegeneracy for real algebraic hypersurfaces and its application to CR mappings, *Math. Z.*, 231, 189-202 (1999).

[NWY] L. Nirenberg, S. Webster, and P. Yang, Local boundary regularity of holomorphic mappings, *Comm Pure Appl Math.*, 33, 305-338 (1980).

[Pi] S. I. Pinchuk, On analytic continuation of biholomorphic mappings, *Mat. USSR Sb.*, 105, 574-594 (1978).

[GR] G. Roberts, Smoothness of CR maps between certain finite type hypersurfaces in complex space, Ph.D. Thesis, Purdue University(1988).

[S] N. Stanton, Infinitesimal CR automorphisms of real hypersurfaces, *Amer. J. of Math*, 118, 209-233 (1996).

[T] F. Treves, Hypo-analytic structures, *Princeton University Press*, Princeton, (1992).

[Tu] A. Tumanov, Analytic discs and the regularity of CR mappings in higher codimension, *Duke Math. J.*, 76, 793-803 (1994).

[W] S. Webster, On mapping an $n-$ball into an $(n+1)-$ball in complex space, *Pac. J. Math*, 81, 267-272 (1979).