A remark on Bergman-Einstein metrics

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Canonical metrics are important objects under study in Complex Analysis of Several Variables. Since Cheng-Yau proved in [CY] the existence of a complete Kähler-Einstein metric over a bounded pseudoconvex domain in $\mathbb{C}^n$ with reasonably smooth boundary, it has become a natural question to understand when the Cheng-Yau metric of a bounded pseudoconvex domain is precisely its canonical Bergman metric. Yau [Y] conjectured that this happens if and only if the domain is a bounded homogeneous domain. As a special case of the Yau's problem, Cheng conjectured in 1979 [Ch] that if the Bergman metric of a smoothly bounded strictly pseudoconvex domain is Kähler-Einstein, then the domain is biholomorphic to the ball.

In this note, we observe an affirmative solution to Cheng's conjecture simply by putting together known results scattered in the literature. (See the two dimensional results obtained by Fu-Wong and Nemirovski-Shafikov [FW] [NS]).

Theorem 0.1. *The Bergman metric of a smoothly bounded strictly pseudoconvex domain is Kähler-Einstein if and only if the domain is biholomorphic to the ball.*

Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) > 0\}$ be a strictly pseudoconvex domain with a smooth defining function ρ. In [Fe1], Fefferman showed that the Bergman kernel function $K(z) = K(z, \overline{z})$ of Ω has the asymptotic expansion

$$K(z) = \frac{\phi(z)}{\rho^{n+1}(z)} + \psi(z) \log \rho(z),$$

where $\phi, \psi \in C^\infty(\overline{\Omega})$ and $\phi|_{\partial\Omega} \neq 0$. In particular, if the boundary $\partial\Omega$ of Ω is spherical, then ψ vanishes to infinite order at the boundary $\partial\Omega$.

We first recall the notion of Fefferman defining functions or Fefferman approximate solutions. Consider the following Monge-Ampére type equation introduced in [Fe2]:

$$J(u) := (-1)^n \det \begin{pmatrix} u & u_{\overline{\beta}} \\ u_\alpha & u_{\alpha\overline{\beta}} \end{pmatrix} = u^{n+1} \det((\log \frac{1}{u})_{\alpha\overline{\beta}}) = 1 \text{ in } \Omega,$$

with $u = 0$ on $b\Omega$. Fefferman proved that for any bounded strictly pseudoconvex domain Ω with smooth boundary, there is a smooth defining function r of Ω such that $J(r) = 1 + O(r^{n+1})$, which

*Supported in part by NSF-1363418.

is called a Fefferman approximate solution or a Fefferman defining function of Ω. Moreover, if $r_1,\ r_2$ are two Fefferman approximate solutions, then $r_1 - r_2 = O(\rho^{n+2})$, where ρ is a given defining function of Ω.

We next recall the Moser normal form theory [CM] and the notion of Fefferman scalar boundary invariants (cf. [Fe3], [G]): Let $M \subset \mathbb{C}^n$ be a real analytic strictly pseudoconvex hypersurface containing $p \in \mathbb{C}^n$. Then there exists a coordinates system $(z, w) := (z_1, ..., z_{n-1}, w)$ such that in the new coordinates, $p = 0$ and M is defined near p by an equation of the form:

$$u = |z|^2 + \sum_{|\alpha|,|\beta|\le 2, l\ge 0} A^l_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta v^l, \tag{1}$$

where $w = u + iv, \alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_n)$ are multiindices. Moreover, the coefficients $A^l_{\alpha\overline{\beta}} \in \mathbb{C}$ satisfies:

- $A^l_{\alpha\overline{\beta}}$ is symmetric with respect to permutation of the indices in α and β, respectively.
- $\overline{A^l_{\alpha\overline{\beta}}} = A^l_{\beta\overline{\alpha}}$.
- $\mathrm{tr} A^l_{2\overline{2}} = 0, \mathrm{tr}^2 A^l_{2\overline{3}} = 0, \mathrm{tr}^3 A^l_{3\overline{3}} = 0$, where $A^l_{p\overline{q}}$ is the symmetric tensor $[A^l_{\alpha\overline{\beta}}]_{|\alpha|=p,|\beta|=q}$ on $\mathbb{C}^{n-1}$ and the traces are the usual tensorial traces with respect to $\delta_{i\overline{j}}$.

Here (1) is called a normal form of M at p. When M is merely smooth, the expansion is in the formal sense. $[A^l_{\alpha\overline{\beta}}]$ are called the normal form coefficients. Recall that a boundary scalar invariant at $p \leftrightarrow 0$, or briefly an invariant of weight $w \ge 0$, is a polynomial P in the normal form coefficients $[A^l_{\alpha\overline{\beta}}]$ of $\partial\Omega$ satisfying certain transformation laws. (See [Fe3] and [G] for more details on this transformation law). Using a Fefferman defining function in the asymptotic expansion of the Bergman kernel function:

$$K(z) = \frac{\phi(z)}{r^{n+1}(z)} + \psi(z) \log r(z), \tag{2}$$

with $\phi, \psi \in C^\infty(\overline{\Omega}),\ \phi|_{\partial\Omega} \ne 0$, then ϕ mod r^{n+1}, ψ mod r^∞ are locally determined. Moreover, if $\partial\Omega$ is in its normal form at $p = 0 \in b\Omega$, then any Taylor coefficient at 0 of ϕ of order $\le n$, and of ψ of any order is a universal polynomial in the normal form coefficients $[A^l_{\alpha\overline{\beta}}]$. (See Boutet-Sjöstrand [BS] and a related argument in [Fe3].) In particular, we state the following result from [C]. (See also [G]):

Proposition 0.2. *([C], [G]) Let Ω be as above and suppose that $\partial\Omega$ is in the Moser normal form up to sufficiently high order. Let r be a Fefferman defining function, and let $\phi,\ \psi$ be as in (2). Then $\phi|_{\partial\Omega} = \frac{n!}{\pi^n}$, $\phi = \frac{n!}{\pi^n} + O(r^2)$ and $P_2 = \frac{\phi - \frac{n!}{\pi^n}}{r^2}|_{\partial\Omega}$ defines an invariant of weight 2 at 0. Furthermore, if $n = 2$, then $P_2 = 0$. If $n \ge 3, P_2 = c_n||A^0_{2\overline{2}}||^2$ for some universal constant $c_n \ne 0$.*

As mentioned earlier, Theorem 0.1 is known in the case of $n = 2$ in [FW] and [NS]. We next assume that $n \geq 3$.

Proof of Theorem 0.1: Recall the Fefferman asymptotic expansion:

$$K(z) = \frac{\phi(z)}{\rho^{n+1}(z)} + \psi(z)\log\rho(z) = \frac{\phi + \rho^{n+1}\psi\log\rho}{\rho^{n+1}} \quad \text{for } z \in \Omega \tag{3}$$

with $\phi, \psi \in C^\infty(\overline{\Omega})$ and $\phi|_{\partial\Omega} \neq 0$, where $\rho \in C^\infty(\overline{\Omega})$ is a smooth defining function of Ω with $\Omega = \{z \in \mathbb{C}^n : \rho(z) > 0\}$. Since $K(z) > 0$ for $z \in \Omega$, we have

$$\phi + \rho^{n+1}\psi\log\rho > 0 \quad \text{for} \quad z \in \Omega. \tag{4}$$

Thus

$$(K)^{-\frac{1}{n+1}}(z) = \frac{\rho}{(\phi + \rho^{n+1}\psi\log\rho)^{\frac{1}{n+1}}} \tag{5}$$

is well-defined in Ω.

Let Ω be a smoothly bounded strongly pseudoconvex domain. We notice that the Kähler-Einstein condition of the Bergman metric is equivalent to the fact that $\log K(z)$ is a Kähler-Einstein potential function of Ω. More precisely, we have $J\left[\left(\frac{\pi^n}{n!}K(z)\right)^{-\frac{1}{n+1}}\right] = 1$ for $z \in \Omega$. (See [FW]). Let $r_0(z) := \left(\frac{\pi^n}{n!}K\right)^{-\frac{1}{n+1}}$. We hence have that $r_0(z) > 0$ and $J(r_0) = 1$ in Ω. We next recall the following computation from [FW]:

Let $\Omega = \{z \in \mathbb{C}^n : \rho > 0\}$ be a bounded strongly pseudoconvex domain with a smooth defining function ρ. If the Bergman metric of Ω is Kähler-Einstein, then the coefficient of the logarithmic term in Fefferman's expansion (3) vanishes to infinite order at $b\Omega$, i.e., $\psi = O(\rho^k)$ for any $k > 0$.

As a consequence, $\phi + \rho^{n+1}\psi\log\rho$ extends smoothly to a neighborhood of $\overline{\Omega}$. Since $\phi|_{\partial\Omega} \neq 0$, we have

$$\phi + \rho^{n+1}\psi\log\rho > 0 \text{ for all } z \in \overline{\Omega}.$$

Hence r_0 extends smoothly to a neighborhood of $\overline{\Omega}$ and it is then easy to conclude that r_0 is a Fefferman defining function of Ω. Then from the way r_0 was constructed, it follows that

$$K(z) = \frac{n!}{\pi^n} r_0^{-(n+1)}. \tag{6}$$

Comparing (6) with (2), we arrive at the conclusion that if we let $r = r_0$ in (2), then $\phi \equiv \frac{n!}{\pi^n}$. Then it follows from Proposition 0.2 that $P_2 = c_n||A^0_{2\bar{2}}||^2 = 0$ at $p \in \partial\Omega$ if $\partial\Omega$ is in the Moser normal form up to sufficiently high order at p with $A^0_{2\bar{2}}$ being the Chern-Moser-Weyl-tensor at p. Consequently, $A^0_{2\bar{2}} = 0$ on $\partial\Omega$, for $c_n \neq 0$. That is, every boundary point of $\partial\Omega$ is a CR umbilical point. By a classical result of Chern-Moser, $\partial\Omega$ is spherical. We then apply the same argument in [NS1] to show Ω is the ball by using a uniformization theorem. To make it easier for the reader, we sketch the proof here. Since $\partial\Omega$ is spherical, then Ω is a quotient of the unit ball by a discrete subgroup of $Aut(\mathbb{B}^n)$. (When $\partial\Omega$ is algebraic, Ω is biholomorphic to the ball by a result in [HJ]. The discussion of the general case is contained in [NS2].) We can thus obtain a complete metric of constant holomorphic sectional curvature on D by descending the standard metric on the ball. By a result of Cheng and Yau [CY], the complete Kähler-Einstein metric on D is unique up to a constant. Thus the Bergman metric of D is proportional to the quotient metric and hence has constant holomorphic sectional curvature. Hence it follows from a well-know result due to Qi-Keng Lu[Lu] that Ω is biholomorphic to the unit ball. ∎

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