Below, “∗” means “turn in”, “no ∗” means “do not turn in, but know how to solve”. If a text is in yellow color, the homework is still at a preliminary stage and might be modified later, but feel free to start working on it. The problems marked “for extra fun” are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration. I will also include an incomplete list of topics.

Topics: Metric space, topology, open sets, closed sets, topological space, examples of topological spaces, continuous function (= map), homeomorphism; constructing new topological spaces: subspace topology, product topology, the topology of disjoint union, quotient topology; manifold, sphere $S^n$, torus $T^2$, projective plane (3 definitions), projective space, examples of surfaces, disk, boundary of a disk, attaching map, cell complex (= CW-complex), weak topology, cellular structures on $S^n$ and $\mathbb{RP}^n$, paths and loops in a topological space, homotopy of maps, path homotopy, path homotopy is an equivalence relation, concatenation of paths, loops, fundamental group, $\pi_1(\mathbb{R}^n)$, ...

Homework 1. Due Friday, January 25.

(1∗) Give three different definitions of the projective plane $\mathbb{RP}^2$ (as in class). Prove that they give the same topological space (i.e. they are homeomorphic). Generalize to $\mathbb{RP}^n$.
(2∗) Prove that $\mathbb{RP}^2$ is a manifold.
(3∗) Let $X$ be the result of collapsing $\partial D^2$ in the disk $D^2$ to a point, with the quotient topology. Prove that $X$ is homeomorphic to $S^2$.
(4) Show that $(-\infty, 0]$ and $\mathbb{R}$ (with their usual topology) are not homeomorphic.
(5) Show that the open unit disc in $\mathbb{R}^n$ (= the interior of $D^n$) is homeomorphic to $\mathbb{R}^n$.
(6) Section 1.1, The fundamental group: Basic constructions, p. 38: # 5*. (Make sure to turn in this problem since it is marked with "∗".)
(7) Show that homotopy of paths is an equivalence relation on the set of paths in a topological space $X$.
(8) Prove that fundamental group is homeomorphism-invariant, i.e. if $(X,x_0)$ and $(Y,y_0)$ are homeomorphic pairs, then $\pi_1(X,x_0) \cong \pi_1(Y,y_0)$.

Topics: The fundamental group of a cartesian product, change of basepoint, simply connected, $S^n$ is simply connected for $n \geq 2$, $\pi_1(S^1)$, connected topological space, path-connected topological space, connected component of a topological space, path-connected component, ...

Homework 2. Due Friday, February 1.

(1∗) Prove that a subset $A$ in a cell complex $X$ (with the weak topology) is open (closed) in $X$ if and only if for each characteristic map $\Phi_i : D^n_i \to X$, $\Phi_i^{-1}(A)$ is open (closed) in $D^n_i$. [See p. 519.]
(2) Trace the definition of cell complex $X$ to define the surjective function $\sqcup_{i,\alpha} D^n_i \to X$. Deduce from the previous exercise that the weak topology on $X$ is the same as the quotient topology induced by this function.
(3∗) Show that any path-connected topological space is connected. Show that if a topological space is connected and locally path-connected, then it is path-connected. (Hint: Use connected components and path-connected components.)
(4*) Prove that a cell complex is connected if and only if it is path connected. [Hint: First show that any cell complex is locally path connected. See p. 523.] More generally, show that for any cell complex \( X \), its connected components and path components agree.

(5) Section 1.1, The fundamental group: Basic constructions, p. 38: \( \# \) 2, 3, 10*, 11, 14*.

(6) Given topological spaces \( X \) and \( Y \), prove that the standard projections \( X \times Y \to X \) and \( X \times Y \to Y \) are continuous.

(7) Learn the proof that \( \pi_1(S^1) \cong \mathbb{Z} \), p. 29-31.

(8*) Is the sphere \( S^2 \) homeomorphic to the torus \( T^2 \)? Generalize to \( S^n \) and \( T^n \).

For extra fun: Describe, as precisely as possible, the fundamental group of the waste basket. (See one in my office.) The same question for the surface of this wastebasket. What is the genus of this surface? (Come to office hours to solve this problem.)

Topics: Induced homomorphism, composition of induced homomorphisms, invariance of \( \pi_1 \) under homeomorphisms, retraction, \( r : X \to A \) and \( r_A : X \to X \), no retraction from \( D^2 \) onto \( \partial D^2 = S^1 \), Brouwer fixed-point theorem, (strong) deformation retraction, deformation retraction implies isomorphism in \( \pi_1 \), Moebius band, manifold with boundary, the fundamental theorem of algebra, homotopy equivalence, contractible space, ...

Homework 3. Due Friday, February 8.

(1) Section 1.1, The fundamental group: Basic constructions, p. 38: \( \# \) 16*, 18*.

(2) Suppose \( r : X \to Y \) is a retraction and \( x_0 \in Y \). Show that the homomorphism \( r_* \) induces by \( r \) on the fundamental groups (at \( x_0 \)) is surjective. If \( \iota : Y \hookrightarrow X \) is the inclusion map, prove that the induced homomorphism \( \iota_* \) is injective.

(3) Prove that if \( Y \) is a deformation retract of \( X \), then \( X \) and \( Y \) are homotopy equivalent.

(4) A topological space \( X \) is contractible if \( X \) is homotopy equivalent to the topological space \( \{pt\} \) consisting of one point. Deduce that if \( X \) deformation retracts to a point, then it is contractible.

(5) About homotopy equivalence and deformation retractions, chapter 0, p. 18: \( \# \) 1, 2, 3*, 5*, 6a*, 6b*. [Hint for problem 5: use the product topology and compactness of the interval \([0, 1]\).] [Hint for problem 6a: use the fact that \([0, 1]\) is connected.]

For fun:

- The Poincaré conjecture says that any closed simply connected 3-dimensional manifold is homeomorphic to \( S^3 \). Find and read its proof. (It is quite hard.) Give a different, shorter proof.
- Pick a particular cell complex \( X \). Deform it until it is unrecognizable to obtain a cell complex \( Y \). Prove that \( X \) is homotopy equivalent to \( Y \). Repeat.

Topics: Invariance of \( \pi_1 \) under basepoint-preserving homotopy equivalence, the zigzag argument for general homotopy equivalence, wedge of (pointed) topological spaces, free group, free product (associativity by representing by permutations), the kernel of a homomorphism, the first isomorphism theorem for groups, van Kampen theorem (proof by switching between parts), applications of the van Kampen theorem: \( \pi_1(S^n) \) for \( n \geq 2 \) (again), the fundamental group of wedge product, presentations of the fundamental groups of closed surfaces, adding 2-cells, adding n-cells for \( n \geq 3 \), presenting any group as the fundamental group of a 2-complex, cell structures on surfaces (closed orientable of genus \( g \), closed non-orientable of genus \( g \)), covering
spaces and subgroups, Hausdorff topological space, locally path-connected spaces, a lift of a map (to a covering space), path lifting property, homotopy lifting property, ...  

**Homework 4. Due Friday, February 15.**

1. Prove if $X$ and $Y$ are path-connected topological spaces and $\varphi : X \rightarrow Y$ is a homotopy equivalence, then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \varphi(x_0))$ is an isomorphism for any choice of $x_0 \in X$. (We did this in class for the case when $\varphi$ is a basepoint-preserving homotopy equivalence; see p. 28 and 37 for an arbitrary homotopy equivalence, but better use the zigzag argument presented in class).

2. Learn the proof of the van Kampen theorem, p. 43-46. The main principle: switching from one part to another. Construct partitions either explicitly or using Lebesgue numbers.

3*. A closed surface is a compact surface without boundary. Describe some examples (at least three) of surfaces that are compact and have boundary. Describe some examples (at least three) of surfaces that are not compact and have no boundary. For each of these examples, show that it is homotopy equivalent to a graph. What are the fundamental groups of these surfaces?

4* Prove in two ways that the fundamental group of a finite connected graph $X$ (= finite path-connected cell complex of dimension at most 1) is a free group. The first way: construct a homotopy equivalence between $X$ and a wedge of finitely many circles. The second way: use the van Kampen theorem directly. (All this can actually be generalized to arbitrary connected graphs.)

5. For this whole course, the numbering and formulations of the problems are meant to be from the online version of the book, as it was at the beginning of this course. The van Kampen theorem: applications to cell complexes, section 1.2, p. 52: # 2, 3*, 4*, 7*, 16*. If you claim that a space is path-connected, provide reasoning.

For fun:

(a) Count the number of cells in each dimension of the standard cellulation of $T^3$. Then do this for $T^n$.

(b) Pick your favorite manifold, construct a cellular structure on it, count the number of cells in each dimension. Find most efficient cellulations of this manifold, or at least as efficient as possible.

(c) Do (a) and (b) for triangulations.

**Know before Friday, February 22.**

1. For this whole course, the numbering and formulations of the problems are meant to be from the online version of the book, as it was at the beginning of this course. The van Kampen theorem: applications to cell complexes, section 1.2, p. 52: # 8. If you claim that a space is path-connected, provide reasoning. [Hint for # 8: use Cartesian products.]

2. Give an explicit description of a covering space of the wedge of two circles, $S^1 \vee S^1$, that is contractible.

3. Covering spaces, section 1.3, p.79: # 1, 2, 3, 4.