Math 525 Algebraic topology I. Spring 2019. Igor Mineyev
Homework, topics, and fun.

Below, “∗” means “turn in”, “no ∗” means “do not turn in, but know how to solve”. If a text is in yellow color, the homework is still at a preliminary stage and might be modified later, but feel free to start working on it. The problems marked “for extra fun” are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration. I will also include an incomplete list of topics.

Topics: Metric space, topology, open sets, closed sets, topological space, examples of topological spaces, continuous function (= map), homeomorphism; constructing new topological spaces: subspace topology, product topology, the topology of disjoint union, quotient topology; manifold, sphere $S^n$, torus $T^2$, projective plane (3 definitions), projective space, examples of surfaces, disk, boundary of a disk, attaching map, cell complex (= CW-complex), weak topology, cellular structures on $S^n$ and $\mathbb{RP}^n$, paths and loops in a topological space, homotopy of maps, path homotopy, path homotopy is an equivalence relation, concatenation of paths, loops, fundamental group, $\pi_1(\mathbb{R}^n)$, ...

Homework 1. Due Friday, January 25.

(1*) Give three different definitions of the projective plane $\mathbb{RP}^2$ (as in class). Prove that they give the same topological space (i.e. they are homeomorphic). Generalize to $\mathbb{RP}^n$.
(2*) Prove that $\mathbb{RP}^2$ is a manifold.
(3*) Let $X$ be the result of collapsing $\partial D^2$ in the disk $D^2$ to a point, with the quotient topology. Prove that $X$ is homeomorphic to $S^2$.
(4) Show that $(-\infty, 0]$ and $\mathbb{R}$ (with their usual topology) are not homeomorphic.
(5) Show that the open unit disc in $\mathbb{R}^n$ (= the interior of $D^n$) is homeomorphic to $\mathbb{R}^n$.
(6) Section 1.1, The fundamental group: Basic constructions, p. 38: # 5*. (Make sure to turn in this problem since it is marked with “∗”.)
(7) Show that homotopy of paths is an equivalence relation on the set of paths in a topological space $X$.
(8) Prove that fundamental group is homeomorphism-invariant, i.e. if $(X, x_0)$ and $(Y, y_0)$ are homeomorphic pairs, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

Topics: The fundamental group of a cartesian product, change of basepoint, simply connected, $S^n$ is simply connected for $n \geq 2$, $\pi_1(S^1)$, connected topological space, path-connected topological space, connected component of a topological space, path-connected component, ...

Homework 2. Due Friday, February 1.

(1*) Prove that a subset $A$ in a cell complex $X$ (with the weak topology) is open (closed) in $X$ if and only if for each characteristic map $\Phi_i : D^n_i \to X$, $\Phi_i^{-1}(A)$ is open (closed) in $D^n_i$. [See p. 519.]
(2) Trace the definition of cell complex $X$ to define the surjective function $\sqcup_{n,i} D^n_i \to X$. Deduce from the previous exercise that the weak topology on $X$ is the same as the quotient topology induced by this function.
(3*) Show that any path-connected topological space is connected. Show that if a topological space is connected and locally path-connected, then it is path-connected. (Hint: Use connected components and path-connected components.)
(4*) Prove that a cell complex is connected if and only if it is path connected. [Hint: First show that any cell complex is locally path connected. See p. 523.] More generally, show that for any cell complex $X$, its connected components and path components agree.

(5) Section 1.1, The fundamental group: Basic constructions, p. 38: # 2, 3, 10*, 11, 14*.

(6) Given topological spaces $X$ and $Y$, prove that the standard projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous.

(7) Learn the proof that $\pi_1(S^1) \cong \mathbb{Z}$, p. 29-31.

(8*) Is the sphere $S^2$ homeomorphic to the torus $T^2$? Generalize to $S^n$ and $T^n$.

For extra fun: Describe, as precisely as possible, the fundamental group of the waste basket. (See one in my office.) The same question for the surface of this wastebasket. What is the genus of this surface? (Come to office hours to solve this problem.)

Topics: Induced homomorphism, composition of induced homomorphisms, invariance of $\pi_1$ under homeomorphisms, retraction, $r : X \to A$ and $r_A : X \to X$, no retraction from $D^2$ onto $\partial D^2 = S^1$, Brouwer fixed-point theorem, (strong) deformation retraction, deformation retraction implies isomorphism in $\pi_1$, Moebius band, manifold with boundary, the fundamental theorem of algebra, homotopy equivalence, contractible space, ...

Homework 3. Due Friday, February 8.

(1) Section 1.1, The fundamental group: Basic constructions, p. 38: # 16*, 18*.

(2) Suppose $r : X \to Y$ is a retraction and $x_0 \in Y$. Show that the homomorphism $r_*$ induces by $r$ on the fundamental groups (at $x_0$) is surjective. If $i : Y \hookrightarrow X$ is the inclusion map, prove that the induced homomorphism $i_*$ is injective.

(3) Prove that if $Y$ is a deformation retract of $X$, then $X$ and $Y$ are homotopy equivalent.

(4) A topological space $X$ is contractible if $X$ is homotopy equivalent to the topological space $\{pt\}$ consisting of one point. Deduce that if $X$ deformation retracts to a point, then it is contractible.

(5) About homotopy equivalence and deformation retractions, chapter 0, p. 18: #1, 2, 3*, 5*, 6a*, 6b*. [Hint for problem 5: use the product topology and compactness of the interval $[0, 1]$.] [Hint for problem 6a: use the fact that $[0, 1]$ is connected.]

For fun:

- The Poincaré conjecture says that any closed simply connected 3-dimensional manifold is homeomorphic to $S^3$. Find and read its proof. (It is quite hard.) Give a different, shorter proof.
- Pick a particular cell complex $X$. Deform it until it is unrecognizable to obtain a cell complex $Y$. Prove that $X$ is homotopy equivalent to $Y$. Repeat.

Topics: Invariance of $\pi_1$ under basepoint-preserving homotopy equivalence, the zigzag argument for general homotopy equivalence, wedge of (pointed) topological spaces, free group, reduced word in the alphabet $\bigcup_a (A_a \setminus \{1\})$, free product (associativity by representing by permutations), the kernel of a homomorphism, the first isomorphism theorem for groups, $j_a : A \to \bigcup_a A_a$, $j_{a\beta} : A_a \cap A_\beta \to A_a$, van Kampen theorem (proof by switching designation between parts), applications of the van Kampen theorem: $\pi_1(S^n)$ for $n \geq 2$ (again), the fundamental group of wedge sum, of a wedge of circles, ...

Homework 4. Due Friday, February 15.
(1) Prove that if \( X \) and \( Y \) are path-connected topological spaces and \( \varphi : X \to Y \) is a homotopy equivalence, then the induced homomorphism \( \varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0)) \) is an isomorphism for any choice of \( x_0 \in X \). (We did this in class for the case when \( \varphi \) is a basepoint-preserving homotopy equivalence; see p. 28 and 37 for an arbitrary homotopy equivalence, but better use the zigzag argument presented in class).

(2) Learn the proof of the van Kampen theorem, p. 43-46. The main principle: switching from one part to another. Construct partitions either explicitly or using Lebesgue numbers.

(3*) A closed surface is a compact surface without boundary. Describe some examples (at least three) of surfaces that are compact and have boundary. Describe some examples (at least three) of surfaces that are not compact and have no boundary. For each of these examples, show that it is homotopy equivalent to a graph. What are the fundamental groups of these surfaces?

(4*) Prove in two ways that the fundamental group of a finite connected graph \( X (= \) finite path-connected cell complex of dimension at most 1) is a free group. The first way: construct a homotopy equivalence between \( X \) and a wedge of finitely many circles. The second way: use the van Kampen theorem directly. (All this can actually be generalized to arbitrary connected graphs.)

(5) For this whole course, the numbering and formulations of the problems are meant to be from the online version of the book, as it was at the beginning of this course.

The van Kampen theorem: applications to cell complexes, section 1.2, p. 52: # 2, 3*, 4*, 7*, 16*. If you claim that a space is path-connected, provide reasoning.

For extra fun:

(a) Count the number of cells in each dimension of the standard cellulation of \( T^3 \). Then do this for \( T^n \).

(b) Pick your favorite manifold, construct a cellular structure on it, count the number of cells in each dimension. Find most efficient cellulations of this manifold, or at least as efficient as possible.

(c) Do (a) and (b) for triangulations.

Topics: Attaching 2-cells to spaces, attaching \( n \)-cells for \( n \geq 3 \), the induced homomorphism of \( X^{(2)} \hookrightarrow X \) (using properties of cell complexes below), fundamental groups of arbitrary complexes, group presentations, presentation complex, presenting any group as the fundamental group of a 2-complex, presentations of the fundamental groups of closed surfaces, connected sum, cell structures on surfaces (closed orientable of genus \( g \), closed non-orientable of genus \( g \)), distinguishing (homotopy types of) closed surfaces by orientation and genus, cell (=open cell), ...

Know before exam on Friday, February 22.

(1) The van Kampen theorem: applications to cell complexes, section 1.2, p. 52: # 8. If you claim that a space is path-connected, provide reasoning. [Hint for # 8: use Cartesian products.]

(2) Prove that any cell complex is Hausdorff (with respect to the weak topology).

(3) Show that for any \( n \)-cell \( e^n \) in a cell complex \( X \), the closure of \( e^n \) in \( X^{(n)} \) is the same as the closure of \( e^n \) in \( X \).

(4) Show that for any cell complex \( X \) and any \( n \), \( X^{(n)} \) is closed in \( X \). Deduce that \( X^{(n)} \) is a subcomplex of \( X \).

For extra fun:
Define the notion of an orientation on a manifold. For Riemannian manifolds, this can be done using the Riemannian structure. For triangulated manifolds, use the simplicial structure. For topological manifolds, use relative singular homology.

**Topics:** Subcomplex (two definitions and their equivalence), compact subsets of cell complexes, finite complex ...

**Homework 5. Due Friday, March 1.**

1* Show that abelianization homomorphisms \( \alpha : G \to G_{ab} \) commute with quotient homomorphisms in the following sense. If \( G \) is a group, \( S \) is a subset of \( G \) and \( \langle S \rangle_G \) is the subgroup of \( G \) normally generated by \( S \), denote \( H := G/\langle S \rangle_G \) and let \( q_S : G \to H \) be the quotient homomorphism. Then the following diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha} & G_{ab} \\
\downarrow q_S & & \downarrow q_{\alpha G(S)} \\
H & \xrightarrow{\alpha_H} & H_{ab}
\end{array}
\]

\( q_{\alpha G(S)} \) here means the quotient map of \( G_{ab} \) by the subgroup normally generated by \( \alpha G(S) \subseteq G_{ab} \). (Since \( G_{ab} \) is abelian, “normally generated” is the same as “generated”.)

2* Let \( F_n \) be the free group of rank \( n \) with basis \( \{x_1, \ldots, x_n\} \). Prove that an element \( x_{i_1}^{m_1} \cdots x_{i_k}^{m_k} \) of \( F_n \), where each \( x_i \) is an element of the basis, belongs to the commutator subgroup \( F_n' \) if and only if, for each \( i \in \{1, \ldots, n\} \), the sum of powers of \( x_i \) occurring in \( x_{i_1}^{m_1} \cdots x_{i_k}^{m_k} \) is zero. (Hint: Use the identity \( ba[a^{-1}, b^{-1}] = ab \).)

3* Use (2) to show that the abelianization of \( F_n \) is \( \mathbb{Z}^n \).

4* Describe how (1) and (3) are useful for computing abelianizations of groups that are given by presentations.

5* Show that for any cell complex \( X \) and any \( n \), \( X^{(n)} \) is closed in \( X \). Deduce that \( X^{(n)} \) is a subcomplex of \( X \).

**Topics:** Covering spaces, a lift of a map (to a covering space), homotopy lifting property, path lifting property, homomorphism induced by a covering, covering spaces and subgroups, lifting criterion for groups, locally path-connected spaces, lifting criterion (for spaces), asphericity of \( S^1 \), the unique lifting property, semi{locally simply connected} space, universal covering, ...

**Homework 6. Due Friday, March 8.**

1) Learn the proof of the homotopy lifting property for arbitrary covering spaces. It is the same as in the proof of the isomorphism \( \pi_1(S^1) \cong \mathbb{Z} \).

2* Give an explicit description of a covering space of the wedge of two circles, \( S^1 \vee S^1 \), that is contractible. Describe the covering projection map and prove that it is indeed a covering space.

3) Find the universal covering of the projective plane \( \mathbb{R}P^2 \). Generalize to \( \mathbb{R}P^n \) for \( n \geq 2 \).

4) Covering spaces, section 1.3, p.79: \# 1, 2*, 3, 4*, 7*, 9*.

5) Learn the proof of the classification theorem for covers, p. 67.

6* Suppose \( \cdot : G \times Z \to Z \), \( (g, z) \mapsto g \cdot z \), is an action of a group on a topological space, an it is continuous in the sense that for each \( g \in G \), the function \( Z \to Z \), \( z \mapsto g \cdot z \) is continuous. Show that each such action gives rise to a group homomorphism \( \lambda : G \to \text{Homeo}(Z) \), and vice versa. Here \( \text{Homeo}(Z) \) is the group of all self-homeomorphisms of \( Z \). (A different, stronger definition of continuity is often used: the function \( \cdot : G \times Z \to Z \) is required to be continuous.)
For extra fun:

- Pick a particular cell complex \( X \), for example a finite graph. Construct a covering \( \tilde{X} \) of this complex. Find a generating set for \( \pi_1(\tilde{X}) \) and describe it in terms of loops in \( \tilde{X} \). Construct another cover of \( X \), find a generating set. Repeat.
- Which 2-dimensional complexes are aspherical?
- Be the first one to prove or disprove the Whitehead conjecture: any subcomplex of any aspherical 2-dimensional cell complex is aspherical.

**Topics:** Existence of universal coverings, existence of a covering for a given subgroup of \( \pi_1(X) \), uniqueness of covering spaces (for a given subgroup), the classification of covering spaces, isomorphism of covering spaces, regular covering (= normal covering), deck transformation (= automorphism of a covering space), group actions, the action of \( G(\tilde{X}) \) on \( \tilde{X} \), the characterization of regular covers (by normality of the subgroup, statement), \( G(\tilde{X}) \cong N(H)/H \) (statement), ...

**Homework 7. Due Friday, March 15.**

1. Find all the deck transformations for the coverings in problems (2) and (3) in the previous homework.
2. Let \( \langle S|R \rangle \) be a presentation of a group \( G \). Consider the following two graphs (= 1-dimensional cell complexes) \( G \) and \( G' \).
   a. The Cayley graph for the generating set \( S \) of \( G \) is the graph \( G \) whose vertices \( v_g \) one-to-one correspond to the elements \( g \in G \) and edges \( e_{g,s} \) one-to-one correspond to the elements \( (g, s) \in G \times S \); the left end of each edge \( e_g \) is attached to \( v_g \) and the right end to \( v_{gs} \).
   b. Let \( X_{S,R} \) be the presentation complex for the presentation \( \langle S|R \rangle \) (that is one vertex, one edge for each \( s \in S \), and one 2-cell for each \( r \in R \)), and let \( G' \) be the 1-skeleton of the universal cover \( \tilde{X}_{S,R} \) of \( X_{S,R} \).
   We proved in class that \( \pi_1(X_{S,R}) \cong G \). (Remember how?) Prove that the graphs \( G \) and \( G' \) are isomorphic (or, equivalently, homeomorphic as cell complexes). [Hint: First extend the Cayley graph \( G \) to the Cayley complex, see page 77. Then use the uniqueness of universal covers.]
3. Covering spaces, section 1.3, p.79: \# 14*, 16*, 17*, 18*, 19*. (In 16, the assumption of being locally path-connected does not seem to be necessary (?), but it is helpful if you want to use the lifting criterion. In 17, make sure to describe \( X \) and \( \tilde{X} \), and check the rest of conditions.)

**Topics:** Deck transformations are uniquely determined by (their value at) one point, the proof of the characterization of regular covers, the proof of \( G(\tilde{X}) \cong N(H)/H \), regular covers arising from group actions, ...

**Topics:** Subgroups of free groups are free. \( \Delta \)-simplex, face, \( \Delta \)-complex, each \( \Delta \)-complex gives rise to a (particular) cell complex (without proof), simplicial complex, a chain complex, the chain complex \( \Delta_*(X) = C_\text{simp}^*(X) \), simplicial homology \( H^n_{\Delta}(X) \) of a \( \Delta \)-complex, ...

**Homework. To know before Exam 2 on April 5.**

1. Learn examples 2.2-2.5 for simplicial homology, p. 106.
2. Simplicial and singular homology, section 2.1, p. 131: \# 4, 5.
3. ...

**Topics:** Exact sequence, short exact sequence, chain map between chain complexes, the long exact sequence corresponding to the short exact sequence of chain complexes, reduced homology. The chain map induced by a map \( X \to Y \) of topological spaces, the map on homology (of
chain complexes) induced by chain maps, the maps induced on (singular) homology by homotopic maps coincide, homotopy equivalent spaces have the same homology, the chain map for a pair \((X, A)\), \(C(X, A)\), the short exact sequence of chain complexes for a pair \((X, A)\), relative homology \(H_\ast(X, A)\), the corresponding long exact sequence (involving \(H_\ast(A), H_\ast(X), H_\ast(X, A)\)), reduced relative homology, the long exact sequence for the reduced relative homology, ...

**Homework 9. Due Friday, April 12.**

1. Prove that each short exact sequence of chain complexes gives rise to a long exact sequence of homology groups. (The best is not to look in the book; do it as an exercise on your own.)

2. Simplicial and singular homology, section 2.1, p. 131: \#7*, 11*, 12*, 13*.

3. ...