Math 525 Algebraic topology I. Spring 2019. Igor Mineyev
Homework, topics, and fun.

Below, "$*$" means "turn in", "no $*$" means "do not turn in, but know how to solve". If a text is in yellow color, the homework is still at a preliminary stage and might be modified later, but feel free to start working on it. The problems marked "**for extra fun**" are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration. I will also include an incomplete list of topics.

Topics: Metric space, topology, open sets, closed sets, topological space, examples of topological spaces, continuous function (= map), homeomorphism; constructing new topological spaces: subspace topology, product topology, the topology of disjoint union, quotient topology; manifold, sphere S^n, torus T^2, projective plane (3 definitions), projective space, examples of surfaces, disk, boundary of a disk, attaching map, cell complex (= CW-complex), weak topology, cellular structures on S^n and $\mathbb{R}\mathrm{P}^n$, paths and loops in a topological space, homotopy of maps, path homotopy, path homotopy is an equivalence relation, concatenation of paths, loops, fundamental group, $\pi_1(\mathbb{R}^n)$, ...

Homework 1. Due Friday, January 25.

(1*) Give three different definitions of the projective plane $\mathbb{R}\mathrm{P}^2$ (as in class). Prove that they give the same topological space (i.e. they are homeomorphic). Generalize to $\mathbb{R}\mathrm{P}^n$.
(2*) Prove that $\mathbb{R}\mathrm{P}^2$ is a manifold.
(3*) Let X be the result of collapsing ∂D^2 in the disk D^2 to a point, with the quotient topology. Prove that X is homeomorphic to S^2.
(4) Show that $(-\infty, 0]$ and $\mathbb{R}$ (with their usual topology) are not homeomorphic.
(5) Show that the open unit disc in $\mathbb{R}^n$ (= the interior of D^n) is homeomorphic to $\mathbb{R}^n$.
(6) Section 1.1, The fundamental group: Basic constructions, p. 38: # 5*. (Make sure to turn in this problem since it is marked with "*".)
(7) Show that homotopy of paths is an equivalence relation on the set of paths in a topological space X.
(8) Prove that fundamental group is homeomorphism-invariant, i.e. if (X, x_0) and (Y, y_0) are homeomorphic pairs, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

Topics: The fundamental group of a cartesian product, change of basepoint, simply connected, S^n is simply connected for $n \geq 2$, $\pi_1(S^1)$, connected topological space, path-connected topological space, connected component of a topological space, path-connected component, ...

Homework 2. Due Friday, February 1.

(1*) Prove that a subset A in a cell complex X (with the weak topology) is open (closed) in X if and only if for each characteristic map $\Phi_i : D^n_i \to X$, $\Phi_i^{-1}(A)$ is open (closed) in D^n_i. [See p. 519.]
(2) Trace the definition of cell complex X to define the surjective function $\sqcup_{n,i} D^n_i \twoheadrightarrow X$. Deduce from the previous exercise that the weak topology on X is the same as the quotient topology induced by this function.
(3*) Show that any path-connected topological space is connected. Show that if a topological space is connected and locally path-connected, then it is path-connected. (Hint: Use connected components and path-connected components.)

(4*) Prove that a cell complex is connected if and only if it is path connected. [Hint: First show that any cell complex is locally path connected. See p. 523.] More generally, show that for any cell complex X, its connected components and path components agree.

(5) Section 1.1, The fundamental group: Basic constructions, p. 38: # 2, 3, 10*, 11, 14*.

(6) Given topological spaces X and Y, prove that the standard projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous.

(7) Learn the proof that $\pi_1(S^1) \cong \mathbb{Z}$, p. 29-31.

(8*) Is the sphere S^2 homeomorphic to the torus T^2? Generalize to S^n and T^n.

For extra fun: Describe, as precisely as possible, the fundamental group of the waste basket. (See one in my office.) The same question for the *surface* of this wastebasket. What is the genus of this surface? (Come to office hours to solve this problem.)

Topics: Induced homomorphism, composition of induced homomorphisms, invariance of π_1 under homeomorphisms, retraction, $r : X \twoheadrightarrow A$ and $r_A : X \to X$, no retraction from D^2 onto $\partial D^2 = S^1$, Brouwer fixed-point theorem, (strong) deformation retraction, deformation retraction implies isomorphism in π_1, Moebius band, manifold with boundary, the fundamental theorem of algebra, homotopy equivalence, contractible space, ...

Homework 3. Due Friday, February 8.

(1) Section 1.1, The fundamental group: Basic constructions, p. 38: # 16*, 18*.

(2) Suppose $r : X \to Y$ is a retraction and $x_0 \in Y$. Show that the homomorphism r_* induces by r on the fundamental groups (at x_0) is surjective. If $\iota : Y \hookrightarrow X$ is the inclusion map, prove that the induced homomorphism ι_* is injective.

(3) Prove that if Y is a deformation retract of X, then X and Y are homotopy equivalent.

(4) A topological space X is *contractible* if X is homotopy equivalent to the topological space $\{pt\}$ consisting of one point. Deduce that if X deformation retracts to a point, then it is contractible.

(5) About homotopy equivalence and deformation retractions, chapter 0, p. 18: #1, 2, 3*, 5*, 6a*, 6b*. [Hint for problem 5: use the product topology and compactness of the interval $[0, 1]$.] [Hint for problem 6a: use the fact that $[0, 1]$ is connected.]

For fun:

- The Poincaré conjecture says that any closed simply connected 3-dimensional manifold is homeomorphic to S^3. Find and read its proof. (It is quite hard.) Give a different, shorter proof.
- Pick a particular cell complex X. Deform it until it is unrecognizable to obtain a cell complex Y. Prove that X is homotopy equivalent to Y. Repeat.

Topics: Invariance of π_1 under basepoint-preserving homotopy equivalence, the zigzag argument for general homotopy equivalence, wedge of (pointed) topological spaces, free group, reduced word in the alphabet $\bigsqcup_\alpha (A_\alpha \setminus \{1\})$, free product (associativity by representing by permutations), the kernel of a homomorphism, the first isomorphism theorem for groups, $j_\alpha : A \to \cup_\alpha A_\alpha$, $j_{\alpha\beta} : A_\alpha \cap A_\beta \to A_\alpha$, van Kampen theorem (proof by switching designation between parts), applications of the van Kampen theorem: $\pi_1(S^n)$ for $n \geq 2$ (again), the fundamental group of wedge sum, of a wedge of circles, ...

Homework 4. Due Friday, February 15.

(1) Prove that if X and Y are path-connected topological spaces and $\varphi : X \to Y$ is a homotopy equivalence, then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism for any choice of $x_0 \in X$. (We did this in class for the case when φ is a *basepoint-preserving* homotopy equivalence; see p. 28 and 37 for an arbitrary homotopy equivalence, but better use the zigzag argument presented in class).

(2) Learn the proof of the van Kampen theorem, p. 43-46. The main principle: switching from one part to another. Construct partitions either explicitly or using Lebesgue numbers.

(3*) A *closed surface* is a compact surface without boundary. Describe some examples (at least three) of surfaces that are compact and have boundary. Describe some examples (at least three) of surfaces that are not compact and have no boundary. For each of these examples, show that it is homotopy equivalent to a graph. What are the fundamental groups of these surfaces?

(4*) Prove *in two ways* that the fundamental group of a finite connected graph X (= finite path-connected cell complex of dimension at most 1) is a free group. The first way: construct a homotopy equivalence between X and a wedge of finitely many circles. The second way: use the van Kampen theorem directly. (All this can actually be generalized to arbitrary connected graphs.)

(5) For this whole course, the numbering and formulations of the problems are meant to be from the *online* version of the book, as it was at the beginning of this course.
The van Kampen theorem: applications to cell complexes, section 1.2, p. 52: # 2, 3*, 4*, 7*, 16*. If you claim that a space is path-connected, provide reasoning.

For extra fun:

(a) Count the number of cells in each dimension of the standard cellulation of T^3. Then do this for T^n.

(b) Pick your favorite manifold, construct a cellular structure on it, count the number of cells in each dimension. Find most efficient cellulations of this manifold, or at least as efficient as possible.

(c) Do (a) and (b) for triangulations.

Topics: Attaching 2-cells to spaces, attaching n-cells for $n \geq 3$, the induced homomorphism of $X^{(2)} \hookrightarrow X$ (using properties of cell complexes below), fundamental groups of arbitrary complexes, group presentations, presentation complex, presenting any group as the fundamental group of a 2-complex, presentations of the fundamental groups of closed surfaces, connected sum, cell structures on surfaces (closed orientable of genus g, closed non-orientable of genus g), distinguishing (homotopy types of) closed surfaces by orientation and genus, cell (=open cell), ...

Know before exam on Friday, February 22.

(1) The van Kampen theorem: applications to cell complexes, section 1.2, p. 52: # 8. If you claim that a space is path-connected, provide reasoning. [Hint for # 8: use Cartesian products.]

(2) Prove that any cell complex is Hausdorff (with respect to the weak topology).

(3) Show that for any n-cell e^n in a cell complex X, the closure of e^n in $X^{(n)}$ is the same as the closure of e^n in X.

(4) Show that for any cell complex X and any n, $X^{(n)}$ is closed in X. Deduce that $X^{(n)}$ is a subcomplex of X.

For extra fun:

- Define the notion of an *orientation* on a manifold. For Riemannian manifolds, this can be done using the Riemannian structure. For triangulated manifolds, use the simplicial structure. For topological manifolds, use relative singular homology.

Topics: Subcomplex (two definitions and their equivalence), compact subsets of cell complexes, finite complex ...

Homework 5. Due Friday, March 1.

(1*) Show that abelianization homomorphisms $\alpha_G : G \to G_{ab}$ commute with quotient homomorphisms in the following sense. If G is a group, S is a subset of G and $\langle\langle S\rangle\rangle_G$ is the subgroup of G normally generated by S, denote $H := G/\langle\langle S\rangle\rangle_G$ and let $q_S : G \to H$ be the quotient homomorphism. Then the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\alpha_G} & G_{ab} \\ \downarrow{\scriptstyle q_S} & & \downarrow{\scriptstyle q_{\alpha_G(S)}} \\ H & \xrightarrow{\alpha_H} & H_{ab} \end{array}$$

$q_{\alpha_G(S)}$ here means the quotient map of G_{ab} by the subgroup normally generated by the subset $\alpha_G(S) \subseteq G_{ab}$. (Since G_{ab} is abelian, "normally generated" is the same as "generated".)

(2*) Let F_n be the free group of rank n with basis $\{x_1, \ldots, x_n\}$. Prove that an element $x_{i_1}^{m_1} \ldots x_{i_k}^{m_k}$ of F_n, where each x_{i_j} is an element of the basis, belongs to the commutator subgroup F_n' if and only if, for each $i \in \{1, \ldots, n\}$, the sum of powers of x_i occurring in $x_{i_1}^{m_1} \ldots x_{i_k}^{m_k}$ is zero. (Hint: Use the identity $ba[a^{-1}, b^{-1}] = ab$.)

(3*) Use (2) to show that the abelianization of F_n is $\mathbb{Z}^n$.

(4*) Describe how (1) and (3) are useful for computing abelianizations of groups that are given by presentations.

(5*) Show that for any cell complex X and any n, $X^{(n)}$ is closed in X. Deduce that $X^{(n)}$ is a subcomplex of X.

Topics: Covering spaces, a lift of a map (to a covering space), homotopy lifting property, path lifting property, homomorphism induced by a covering, covering spaces and subgroups, lifting criterion for groups, locally path-connected spaces, lifting criterion (for spaces), asphericity of S^1, the unique lifting property, semi{locally simply connected} space, universal covering, ...

Homework 6. Due Friday, March 8.

(1) Learn the proof of the homotopy lifting property for arbitrary covering spaces. It is the same as in the proof of the isomorphism $\pi_1(S^1) \cong \mathbb{Z}$.

(2*) Give an explicit description of a covering space of the wedge of two circles, $S^1 \vee S^1$, that is contractible. Describe the covering projection map and prove that it is indeed a covering space.

(3) Find the universal covering of the projective plane $\mathbb{R}\mathrm{P}^2$. Generalize to $\mathbb{R}\mathrm{P}^n$ for $n \geq 2$.

(4) Covering spaces, section 1.3, p.79: # 1, 2*, 3, 4*, 7*, 9*.

(5) Learn the proof of the classification theorem for covers, p. 67.

(6*) Suppose $\cdot : G \times Z \to Z$, $(g, z) \mapsto g \cdot z$, is an action of a group on a topological space, an it is *continuous* in the sense that for each $g \in G$, the function $Z \to Z$, $z \mapsto g \cdot z$ is continuous. Show that each such action gives rise to a group homomorphism $\lambda : G \to Homeo(Z)$, and vice versa. Here $Homeo(Z)$ is the group of all self-homeomorphisms of Z. (A different, stronger definition of continuity is often used: the function $\cdot : G \times Z \to Z$ is required to be continuous.)

For extra fun:

- Pick a particular cell complex X, for example a finite graph. Construct a covering $\tilde{X}$ of this complex. Find a generating set for $\pi_1(\tilde{X})$ and describe it in terms of loops in $\tilde{X}$. Construct another cover of X, find a generating set. Repeat.
- Which 2-dimensional complexes are aspherical?
- Be the first one to prove or disprove the Whitehead conjecture: any subcomplex of any aspherical 2-dimensional cell complex is aspherical.

Topics: Existence of universal coverings, existence of a covering for a given subgroup of $\pi_1(X)$, uniqueness of covering spaces (for a given subgroup), the classification of covering spaces, isomorphism of covering spaces, regular covering (= normal covering), deck transformation (= automorphism of a covering space), group actions, the action of $G(\tilde{X})$ on $\tilde{X}$, the characterization of regular covers (by normality of the subgroup, statement), $G(\tilde{X}) \cong N(H)/H$ (statement), ...

Homework 7. Due Friday, March 15.

(1) Find all the deck transformations for the coverings in problems (2) and (3) in the previous homework.

(2*) Let $\langle S|R\rangle$ be a presentation of a group G. Consider the following two graphs (= 1-dimensional cell complexes) $\mathcal{G}$ and $\mathcal{G}'$.

   (a) The *Cayley graph* for the generating set S of G is the graph $\mathcal{G}$ whose vertices v_g one-to-one correspond to the elements $g \in G$ and edges $e_{g,s}$ one-to-one correspond to the elements $(g, s) \in G \times S$; the left end of each edge e_g is attached to v_g and the right end to v_{gs}.

   (b) Let $X_{S,R}$ be the presentation complex for the presentation $\langle S|R\rangle$ (that is one vertex, one edge for each $s \in S$, and one 2-cell for each $r \in R$), and let $\mathcal{G}'$ be the 1-skeleton of the universal cover $\tilde{X}_{S,R}$ of $X_{S,R}$.

   We proved in class that $\pi_1(X_{S,R}) \cong G$. (Remember how?) Prove that the graphs $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic (or, equivalently, homeomorphic as cell complexes). [Hint: First extend the Cayley graph $\mathcal{G}$ to the *Cayley complex*, see page 77. Then use the uniqueness of universal covers.]

(3) Covering spaces, section 1.3, p.79: # 14*, 16*, 17*, 18*, 19*. (In 16, the assumption of being locally path-connected does not seem to be necessary (?), but it is helpful if you want to use the lifting criterion. In 17, make sure to describe X and $\tilde{X}$, and check the rest of conditions.)

Topics: Deck transformations are uniquely determined by (their value at) one point, the proof of the characterization of regular covers, the proof of $G(\tilde{X}) \cong N(H)/H$, regular covers arising from group actions, ...

Topics: Subgroups of free groups are free. Δ-simplex, face (two interpretations: set and map), restriction to a face, Δ-complex, each Δ-complex gives rise to a (particular) cell complex (without proof), simplicial complex, the standard simplex Δ^n, face maps $\Delta^{n-1} \to \Delta^n$, taking the boundary of a manifold twice, the chain complex $\Delta_*(X) = C_*^{simpl}(X)$, $\partial \circ \partial = 0$, simplicial homology $H_n^{\Delta}(X)$ of a Δ-complex, ...

Homework. To know before Exam 2 on April 5.

(1) Learn examples 2.2-2.5 for simplicial homology, p. 106.

(2) Simplicial and singular homology, section 2.1, p. 131: # 4, 5.

Topics: Singular homology $H_n(X) = H_n^{sing}(X)$ of a topological space X, reduced singular homology for a nonempty X, a chain complex, cycles and boundaries, exact sequence, short exact sequence, chain map between chain complexes, the long exact sequence corresponding to a short exact sequence of chain complexes (hw), the chain complex $C_*^{sing}(X)$, the chain map f_* induced by a map $f : X \to Y$ of topological spaces, the map $\bar{f}_*$ on homology (of chain complexes) induced by a chain map f_*, chain homotopy between chain maps, ...

Homework 8. Due Friday, April 12.

(1*) Prove that each short exact sequence of chain complexes gives rise to a long exact sequence of homology groups. (The best is not to look in the book, at least first; do it first as an exercise on your own.)

(2*) Compute the (singular) homology groups of a point. Compute the reduced homology groups of a point.

(3) Simplicial and singular homology, section 2.1, p. 131: #7*, 11*, 13*. (For # 7 make an educated guess how to glue the 3-simplex to obtain the 3-sphere, without proof. Then compute simplicial homology. For extra fun: Prove that the result of gluing is indeed *topologically* the 3-sphere. Also, is the result of this (combinatorial) gluing a Δ-complex? Is it a simplicial complex?)

(4) Compute the 0th (singular) homology of any topological space X, $H_0(X, \mathbb{Z})$. (See Proposition 2.7, p. 109.)

Topics: The statement for the long exact sequence for a good pair (X, A) (involving $\tilde{H}_*(A)$, $\tilde{H}_*(X)$, $\tilde{H}_*(X/A)$), the chain complex for a pair (X, A), $C(X, A)$, the short exact sequence of chain complexes for (any) pair (X, A), relative homology $H_*(X, A)$, the corresponding long exact sequence (involving $H_*(A)$, $H_*(X)$, $H_*(X, A)$), reduced relative homology $\tilde{H}_*(X, A)$, $\tilde{H}_*(X, A) \cong H_*(X, A)$, the long exact sequence for reduced relative homology, $f \rightsquigarrow f_* \rightsquigarrow \bar{f}_*$, $f \sim g \Rightarrow f_* \sim g_* \Rightarrow \bar{f}_* = \bar{g}_*$, the maps induced on (singular) homology by homotopic maps coincide, chain homotopy equivalence of chain complexes, $X \sim Y \Rightarrow C_*(X) \sim C_*(Y) \Rightarrow H_*(X) \cong H_*(Y)$, homotopy equivalent spaces have the same homology, the statement of the excision theorem, the cone-off map $b : LC_n(Y) \to LC_{n+1}(Y)$ (for $b \in Y$, Y convex in $\mathbb{R}^k$), the barycenter of a linear simplex, the barycentric subdivision of a linear simplex, ...

Homework 9. Due Friday, April 19.

(1) Simplicial and singular homology, section 2.1, p. 131: # 12*, 23*.

(2*) In your own words, write a detailed proof that $H_1(X)$ is (isomorphic to) the abelianization of $\pi_1(X)$ for path connected X. See p. 166. This is part of what is known as the Hurewicz theorem.

Topics: An outline of the proof of the excision theorem, $H_*^{\mathcal{U}}(X) \cong H_*(X)$ (i.e. $H_*(X)$ is isomorphic to the homology using arbitrarily small singular simplices in X), the barycentric subdivision of a linear chain (in $LC_n(Y)$), the barycentric subdivision of a singular chain (in $C_n(X)$), the cone-off map $b : LC_n(Y) \to LC_{n+1}(Y)$ is a contracting homotopy of $LC_*(Y)$ (i.e. a chain homotopy between id_* and 0_*), the subdivision maps $S_* : LC_*(Y) \to LC_*(Y)$ and $S_* : C_*(X) \to C_*(X)$ are chain maps, the chain homotopy T_* between S_* and $id_* : LC_*(Y) \to LC_*(Y)$, the chain homotopy T_* between S_* and $id_* : C_*(X) \to C_*(X)$, the chain homotopy between S_*^m and id_*, subdivision of linear simplices shrinks the diameter by $\frac{n}{n+1}$, the Lebesque number of an open cover (of a compact metric space), the long exact sequence for triples (X, A, B), the proof of the long exact sequence for a good pair (X, A) (involving $\tilde{H}_*(A)$, $\tilde{H}_*(X)$, $\tilde{H}_*(X/A)$), homology of X decomposes as sum over path-connected components, homology decomposes over path-connected components, homology of the sphere S^n (hw), ...

Homework 10. Due Friday, April 26.

(1) Prove that for any family of spaces $\{X_i \mid i \in I\}$, $H_n(\sqcup_i X_i) \cong \oplus_i H_n(X_i)$. (Here $\sqcup_i X_i$ is the formal disjoint union of the spaces X_i. This statement can be viewed as a special case of Proposition 2.6, p. 109.)

(2*) Let x_0 be a point (called a base point) in a topological space X. Prove that $\tilde{H}_n(X) \cong H_n(X, x_0)$. (These are the reduced homology of X and the relative homology of the pair $(X, \{x_0\})$, respectively. See Example 2.18, p. 118. Write a detailed proof.)

(3*) Compute the reduced homology of the sphere S^n, $\tilde{H}_i(S^n)$, in each dimension $i \in \mathbb{Z}$. (See Corollary 2.14, p. 114. Write a detailed proof.)

(4*) Write a detailed proof of Corollary 2.25, p. 126. (You can use Proposition 2.22 that we will prove in class). Then use it to compute the reduced homology of any wedge of n-spheres, $\vee_{j \in J} S_j$. (That is, each S_j is homeomorphic to the n-sphere S^n for the same n, and the problem is to compute $\tilde{H}_i(\vee_j S_j)$ in each dimension i. You can use (1) for this problem.)

(5) Simplicial and singular homology, section 2.1, p. 131: # 22*. (In this problem, "free" should be interpreted as "free abelian". You can use the result from algebra: any subgroup of a free abelian group is free abelian.)

(6) Learn the definition of cellular homology and the statement of the cellular boundary formula, p. 139-140.

(7*) Write a detailed computation of the cellular homology of (particular cellular structures of) closed surfaces. (See particular cellular structures in examples 2.36 and 2.37, p. 141. Use the cellular boundary formula.)

Topics: Subcomplexes form good pairs (without proof), the degree of a map $S^n \to S^n$, no retraction from D^n to ∂D^n, the Brouwer fixed-point theorem for D^n, homology of wedge sum (hw), 5-lemma, $H_n(D^n, \partial D^n)$, simplicial and singular homology are isomorphic, definition of cellular homology $H_n^{cell}(X)$ (hw), cellular boundary formula (without proof), the Mayer-Vietoris sequence, the Euler characteristic of a finite cell complex, homology with coefficients (for example, $\mathbb{R}$ or $\mathbb{C}$), relation to analysis, Hilbert spaces, etc, analytic ways to compute Euler characteristic, ...

To know before the final exam.

(1) Learn the definition and properties of degree, p. 134 and Theorem 2.28 (tangent vector fields on spheres).

(2) First describe the "natural" cellular structures on the complex projective space $\mathbb{C}\mathrm{P}^n$ and on the real projective space $\mathbb{R}\mathrm{P}^n$. Then compute their homology $H_i(\mathbb{C}\mathrm{P}^n)$ and $H_i(\mathbb{R}\mathrm{P}^n)$. (See pp. 140 and 144.)

(3) Prove the 5-lemma without looking in the book.

(4) Homology: Computations and applications, section 2.2, p. 155: #20, 21, 22.