Math 403 Euclidean geometry, Spring 2019. Igor Mineyev.
Homework, topics, practice test, fun.

Below, "$*$" means "*turn in*", "no $*$" means "*do not turn in, but know how to solve*". If a homework is in yellow color, it is still at a preliminary stage and might be modified later, but feel free to start working on it. I will also include an incomplete list of topics. Your practice test consists of:

- homework (listed below),
- topics (listed below),
- your class notes, and
- the textbook material covered in class.

Use them all as a guide to prepare for exams. If any questions arise, please ask me. Coming to office hours is also a very good idea. The problems marked "**for extra fun**" are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration.

Topics: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, set, function, equality of sets, equality of functions, equivalence of statements, cartesian product, $\mathbb{R}^2$, the plane, a point in the plane, vector, group, abelian group, $\mathbb{R}^{100}$, injective, surjective, bijective, the standard (or abstract) line, the line passing through two (distinct!) points, equation of a line, equality of sets, equality of vectors, equality of functions, $\vec{AB}$, ...

Homework 1. Due Friday, January 25, 2019.

(1*) Prove formally and rigorously that the eight properties of vectors in the plane stated in class indeed hold. Those are named (A1), (A2), (A3), (A4), (M1), (M2), (M3), (M4) in the book.

(2*) (a) Write formally what it means for two sets A and B *not* to be equal.
(b) Write formally what it means for two functions $f : A \to B$ and $f' : A' \to B'$ *not* to be equal.
(c) First *define* what it means for two vectors $X = \langle x_1, x_2 \rangle$ and $Y = \langle y_1, y_2 \rangle$ to be the same, i.e. $X = Y$. Then state what it formally means for these two vectors to be different, i.e. $X \neq Y$.

(3) Prove that $(\mathbb{R}^2, +)$ is a group. (We assume it to be known that $(\mathbb{R}, +)$ is a group.)

(4) In analogy with problem (1), state eight properties for $\mathbb{R}^{100}$ instead of $\mathbb{R}^2$. Do these properties hold? Provide at least some justification for the answer.

(5) Chapter 1, page 5, exercises 1.2*, 1.3*, 1.4*, 1.5*. Make sure that you *only* use the eight properties of $\mathbb{R}^2$. For these problems, do *not* write vectors as pairs of numbers. (Make sure to turn all the problems marked with "*".)

For extra fun.

- How to get something out of nothing: Prove that the sets $\emptyset$ and $\{\emptyset\}$ are not equal.
- Prove that any two sets from the list

$$\emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\{\{\emptyset\}\}\},\ \{\{\{\{\emptyset\}\}\}\},\ \{\{\{\{\{\emptyset\}\}\}\}\},\ \{\{\{\{\{\{\emptyset\}\}\}\}\}\},\ \ldots$$

are not equal.

- Prove that "the cartesian product of any two groups is a group". It is also called *the direct product* of the two groups.
  More precisely, let $(G, \cdot)$ and $(H, \cdot)$ be groups. (These two dots are meant to be different operations. To be precise, we should have denoted them, say, $\cdot_G$ and $\cdot_H$, but we won't bother.) Define a "reasonable" binary operation $\cdot$ on $G \times H$ (still another operation which we could as well denote $\cdot_{G\times H}$) so that $(G \times H, \cdot)$ becomes a group. Prove that $(G \times H, \cdot)$ is indeed a group.
- Find your favorite group. Investigate what properties it has. Let me know what you found. Repeat.

Topics: General definition of proportionality of vectors, definition of a nonzero vector, proportionality of nonzero vectors, parallel lines, a basis in $\mathbb{R}^2$, $[A, B]$, collinear points, triangle, parallelogram, an equivalent description of a line, midpoint, points *between* A and B, median of a triangle, medians have a common point, intersection of sets, union of sets, A and $\{A\}$, centroid(= center of mass) of a system of points, intersection and union of sets, ...

Homework 2. Due Friday, February 1.

A general requirement for this class: whenever you divide by a number, say, a, first explain why this number a is not zero. Whenever you use the notation l_{AB}, first explain why $A \neq B$.

(1) Prove that for two nonzero vectors C and D, the two definitions of proportionality given in class are equivalent.

(2*) Let A and B be distinct points in the plane and define

$$l_{AB} := \{A + t(B - A) \mid t \in \mathbb{R}\} = \{P \in \mathbb{R}^2 \mid \exists t \in \mathbb{R} \quad P - A = t(B - A)\}.$$

Let C and D be two distinct points on the line l_{AB}. Prove that $l_{AB} = l_{CD}$. (l_{AB} and l_{CD} are *sets*. What does it mean for two sets to be equal?)

(3*) Let $l = l_{AB}$ be a line and C be a point in the plane not on the line l. Let C' be any point in l. Prove that the intersection $l_{AB} \cap l_{CC'}$ consists of exactly one point. What is that point?

(4*) For any triangle $\{A, B, C\}$, let A', B', C' be the midpoints of the sides $[B, C]$, $[A, C]$, $[A, B]$, respectively. Show that the lines $l_{AA'}$, $l_{BB'}$, $l_{CC'}$ are well-defined. (There are many ways to prove this. The simplest argument seems to be similar to the previous exercise.)

(5*) Prove that for any two vectors $B, C \in \mathbb{R}^2$, given explicitly as *columns* $B = (b_1, b_2)$ and $C = (c_1, c_2)$, the following statements are equivalent.
  (a) $\{B, C\}$ is a basis of $\mathbb{R}^2$.
  (b) Each equation of the form $A = x_1 B + x_2 C$ has a unique solution (x_1, x_2).
  (c) $b_1 c_2 - b_2 c_1 \neq 0$.

(This exercise does not assume any knowledge of linear algebra. If you use, say, Cramer's rule or any other statement from linear algebra, prove it first.)

(6*) Prove that for $A, B, C \in \mathbb{R}^2$, the following statements are equivalent.
  (a) A, B, C are not collinear.
  (b) $\{\vec{AB}, \vec{AC}\}$ is a basis of $\mathbb{R}^2$.

(c) $\{\vec{BA}, \vec{BC}\}$ is a basis of $\mathbb{R}^2$.

Topics: Centroid of three points $\{A, B, C\}$ and the medians $l_{AA'}$, $l_{BB'}$, $l_{CC'}$, centroid of three points and the medians, relation between the centroid of a system of points and the centroids of its two parts, also of three parts, mass-point, centroid of a system of mass-points, points in l_{AB} as centroids of two mass-points (non-uniquely), barycentric coordinates, sufficient condition for (arbitrary) lines $l_{AA'}$, $l_{BB'}$, $l_{CC'}$ to have a common point, when and how the ratio of vectors is defined, ...

Homework 3. Due Friday, February 8.

(1*) For any triangle $\{A, B, C\}$, let A', B', C' be the midpoints of the sides $[B, C]$, $[A, C]$, $[A, B]$, respectively. We showed in the previous homework that the lines $l_{AA'}$, $l_{BB'}$, $l_{CC'}$ are well-defined. And we proved in class that the medians $l_{AA'}$, $l_{BB'}$, $l_{CC'}$ have a common point. Show that this point is unique. [Hint: Consider, say, $C - G$ and $B - G$, where G is the centroid of the triangle, and prove first that they are not proportional.]

(2*) Let A and B be distinct points in the plane and C and D be distinct points in the plane. Prove that the following statements are equivalent.
  (a) The lines l_{AB} and l_{CD} intersect at exactly one point.
  (b) The vectors $\vec{AB}$ and $\vec{CD}$ are not proportional.

(3*) Let l_{AB} and l_{CD} be lines in the plane. Prove that exactly one of the following conditions must hold.
  (a) The two lines do not intersect, i.e. $l_{AB} \cap l_{CD}$ is the empty set.
  (b) The intersection $l_{AB} \cap l_{CD}$ has exactly one point.
  (c) $l_{AB} = l_{CD}$.

(4) Pages 7- 9, exercises 1.6, 1.7*, 1.8*.

(5) Page 9, exercise 1.9*. Compare the centers of $M_1M_2M_3M_4$ and $N_1N_2N_3N_4$.

(6) Page 11, exercise 1.10*, 1.14.

For extra fun.

- Define what should be called barycentric coordinates in $\mathbb{R}^{100}$. State and prove a theorem about their existence and uniqueness. What should be the assumptions in that theorem?

Topics: The theorem of Ceva (complete and detailed statement and proof), ...

Homework. To know before exam on Friday, February 15.

(1) Given three points A, B, C in the plane with $B \neq C$, write an equation of the line l passing through A and parallel to the line l_{BC}.

(2) Let B and C be *distinct* points in the plane. The interval (B, C) is defined as the set

$$(B, C) := \{B + t(C - B) \mid t \in (0, 1)\}.$$

  (a) Suppose $A' \in (B, C)$, $A' = bB + cC$, and $b + c = 1$. Prove that $b, c \in (0, 1)$.
  (b) Suppose $b, c \in (0, 1)$, $b + c = 1$, and $A' = bB + cC$. Prove that $A' \in (B, C)$.

Topics: The theorem of Menelaus (complete, detailed proof that does not rely on the theorem of Ceva), ...

Homework 4. Due Friday, February 22.

(1*) Two lines l_{AB} and l_{CD} are called *parallel* if the (nonzero) vectors $B - A$ and $D - C$ are proportional. Show that this notion of being parallel is well defined, i.e. it depends only on the lines, and not on particular points on the lines. Specifically, prove that if $A', B' \in l_{AB}$ are distinct points and $C', D' \in l_{CD}$ are distinct point, then the vectors $B - A$ and $D - C$ are proportional if and only if $B' - A'$ and $D' - C'$ are proportional.

(2*) Suppose that lines l_{AB} and l_{CD} are parallel and have a common point P. Prove that $l_{AB} = l_{CD}$.

(3) Page 11, exercises 1.11*, 1.12*, 1.13*.

For extra fun.

- The game Euclidea at `www.euclidea.xyz`. It keeps you busy drawing tons of lines, circles, shapes in the plane. The game does not ask you to *prove* anything, so it is not a replacement for a course. But it might help you explore shapes, come up with your own conjectures, and to guess how to prove them.

Below are just some randomly chosen topics that you might consider for your project, they are intended as a guide only. Do your own research, suggest your own topic. Find what you like. Discuss your possible topic with me in advance.

(1) Conformal transformations of the plane, of the disc, of the half-space.
(2) What is the shape of the universe?
(3) Lorentzian geometry, space-time.
(4) Knot theory, a part of topology.
(5) Topology of manifolds, geometry of manifolds.
(6) Many kinds of constructions with multiple triangles, lines and circles.
(7) Königsberg bridges, graph theory, topology.
(8) Flat spaces, negative curvature, positive curvature. Is the universe curved?
(9) Group actions on various geometric objects.
(10) Graphs, groups, Cayley graphs, how to visualize groups geometrically.
(11) Geometry in art, geometric patterns and their mathematical meaning.
(12) The fundamental group of a topological space, definition, many examples.
(13) Patterns, tessellations of the plane.
(14) Inner product on a vector space (aka dot product), Hilbert spaces.
(15) Riemannian structure, Riemannian manifolds, tangent vectors, the intrinsic metric (= path metric) on a manifold.
(16) Euclidea game project? `www.euclidea.xyz`. Make sure that it consists not only of pictures, there must be a true mathematical component in it.
(17) What is geometric group theory? How it relates to other areas of mathematics?
(18) We will talk in class about translations, central dilatations, and other transformations of the plane. How are they related to transformations of the sphere? How are they related to Möbius transformations? To get an idea, watch this video first: `https://www.youtube.com/watch?v=G87ehdmHeac` And read the related article: `http://www-users.math.umn.edu/~arnold//papers/moebius.pdf`
(19) Affine transformations is another class of transformations of the plane $\mathbb{R}^2$, and more generally, of $\mathbb{R}^n$.

(20), (21), (22), ...

Topics: The theorem of Desargues (complete statement for now), the theorem of Pappus (complete statement for now), informal discussion of transformations of objects (Möbius transformations, affine transformations), groups, their possible use for Euclidean geometry, isometries, composition, inverse function, translation τ_A, central dilatation δ_r, $\delta_{C,r}$, the symmetric group $S(V)$ (hw), subgroup, ...

Homework 5. Due Friday, March 1.

(1) Page 13, exercises 1.15, 1.16*, 1.17, 1.18*, 1.20. (In 1.17, it is an equality of sets.)

(2) Let $\{A, B, C\}$ be a triangle (in general position, as always), $A' \in (B, C)$, $B' \in (A, C)$, $C' \in (A, B)$. Prove that

$$\frac{|A' - B|}{|A' - C|}\frac{|B' - C|}{|B' - A|}\frac{|C' - A|}{|C' - B|} = 1.$$

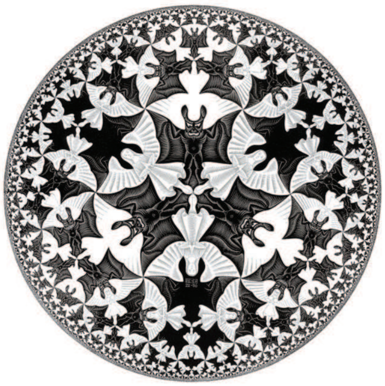
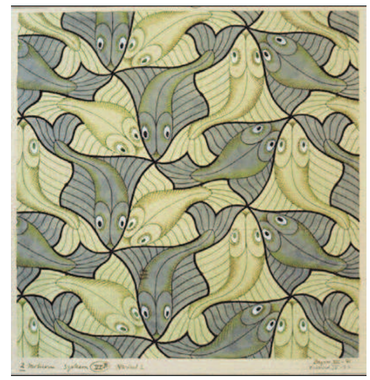
Here $|X|$ denotes the length of a vector X. This is a weaker version of the theorem of Ceva, and it follows from the theorem of Ceva proved in class. In this exercise, do *not* use the theorem of Ceva proved in class. Rather, assume the formula for the area of a triangle, and use areas of triangles for the proof. [See p. 25.]

(3) Page 28, exercise 1.22*, 1.23*, 1.24*. (In 1.24, the theorem of Desargues is proved under the additional assumption that all the lines are pairwise transversal, i.e. each pair of lines intersects at exactly one point.)

Topics: Images of lines under translations and central dilatations, relations between translations and central dilatations, subgroup test, the (sub)group of translations of the plane (hw), a group of transformations of the plane (different from the book), ...

For extra fun.

- Let Isom($\square$) be the set of isometries of the square. Is Isom($\square$) a group (in the sense of the formal definition of a group)? What is the operation? Is this group abelian?
- We know the answer to the question: given a line l in $\mathbb{R}^2$ and a point $A \in \mathbb{R}^2$, how many lines are there in $\mathbb{R}^2$ that pass through A and are parallel to l? The answer to this question is the property called *Euclid's fifth postulate*. Define your own "universe" (the formal term is a *metric space*) that has lines and define a notion of parallelism in such a way that Euclid's fifth postulate does *not* hold. Such spaces go under various names: hyperbolic space, hyperbolic geometry, non-Euclidean space. An example is the hyperbolic plane $\mathbb{H}^2$; on the left is its artistic rendering by Escher (see www.mcescher.com/gallery/)

- Let G be the set of all bijections $\mathbb{H}^2 \to \mathbb{H}^2$ that map angels to angels and devils to devils. (We also require the bijections to be continuous both ways.) Prove that G is a group. Is G abelian? What other interesting properties does G have?
- Let X be the Escher's picture on the right. What kind of geometry does it represent? Let G be the set of all bijections $X \to X$ continuous both ways that map each fish to a fish of the same color. Prove that G is a group. Is G abelian? What other interesting properties does G have?
- Pick your favorite graph Γ; it consists of vertices and edges attached to vertices. Let H be the set of all bijections $\Gamma \to \Gamma$ that map the vertices onto the vertices and edges onto the edges, and preserve the structure of the graph Γ. Prove that H is a group with respect to the composition operation. Investigate, how many elements H has. Is H abelian? Does H have any other interesting properties?

Homework 6. Due Friday, March 8.

(1) Page 34, exercise 2.1*.
(2*) Prove that for any bijection $f : A \to B$ there exists another function $g : B \to A$ such that for any $a \in A$, $(g \circ f)(a) = a$, and for any $b \in B$, $(f \circ g)(b) = b$. (This function g is called the *inverse function* of f and is also denoted f^{-1}.)
(3*) Let V be any set and $S(V)$ be the set of all bijections from V to itself. Prove that $(S(V), \circ)$ is a group. This group is called the *symmetric group* of the set V.
(4*) Prove that the set of all translations, $\{\tau_A : \mathbb{R}^2 \to \mathbb{R}^2 \mid A \in \mathbb{R}^2\}$, is a subgroup of $S(\mathbb{R}^2)$. (This implies, in particular, that the set of all translations is a group itself.)
(5*) Prove that the set of all central dilatations *centered at the origin*
$$\{\delta_r : \mathbb{R}^2 \to \mathbb{R}^2 \mid r \in \mathbb{R} \setminus \{0\}\},$$
is a subgroup of $S(\mathbb{R}^2)$.

For extra fun. *Prove* the following subgroup test theorem:
If $(G, \cdot)$ is a group and H is a subset of G, then the following two statements are equivalent.

(1) H is a subgroup of G.
(2) (a) H is nonempty,
 (b) $\forall\, x, y \in H \quad x \cdot y \in H$, and
 (c) $\forall\, x \in H \quad x^{-1} \in H$.

(Here $\cdot$ is the operation in G and x^{-1} denotes the inverse of x *in* G.)

Topics: Relations between translations and central dilatation, dilatation, the (sub)group of dilatations, ...

Homework. To know before exam, Friday, March 15.

(1) Let α be a dilatation (i.e. translation or a central dilatation) and l_{PQ} be any line.
   (a) Prove that $\alpha(l_{PQ}) = l_{\alpha(P),\alpha(Q)}$. (Equality of sets.)
   (b) Deduce that $\alpha(l_{PQ})$ is a line.
   (c) Prove that this line $\alpha(l_{PQ})$ is parallel to l_{PQ}.
(2) Page 37, exercises 2.2, 2.3.
(3) Page 46, exercises 2.12, 2.13, 2.15, 2.16, 2.17.

Topics: Dot product (= scalar product, inner product) and its properties, the length of a vector, distance in the plane (again), orthogonal vectors (= perpendicular vectors), ...

The project schedule:

- Group 1: Monday, April 22.
- Group 2: Wednesday, April 24.
- Group 3: Friday, April 26.

For extra fun.

- Look up the definition of the semidirect product of two groups. Is the group of dilatations, $D(\mathbb{R}^2)$, a semidirect product in any way?
- Look up the definition of a group action on a set. Does the group $D(\mathbb{R}^2)$ act on anything in any way?

Homework 7. Due Friday, March 29.

(1) Page 39, exercises 2.4*, 2.5*, 2.6*.
(2) Page 46, exercises 2.14*.
(3) List all the permutations of the set $\{A, B, C, D\}$ that correspond to rigid motions of the square.
(4) Define the permutations σ and ρ as on page 55. Explicitly describe each of the following permutations: $id, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3$. Here "product" means composition; say, $\sigma\rho^2$ means $\sigma \circ \rho \circ \rho$. How do these permutations compare to those in problem (3)?

Topics: Orthogonal lines (= perpendicular lines), the theorem of Pythagoras, perpendicular bisector (definition and equivalent restatement), perpendicular bisectors (a, b, c) in a triangle, circle $\mathcal{C}(D, r)$, circumcenter K, the existence of circumcenter, altitudes (l_A, l_B, L_C), images of circles under dilatations, orthocenter, the existence of orthocenter, Euler line, images of circles under dilatations, the nine-point theorem, ...

For extra fun.

- List all the permutations corresponding to all the rigid motions of an equilateral triangle. Define two permutations σ and ρ, and make a list as in problem (4) above.
- Do the same for a regular pentagon, hexagon, etc.
- Describe all groups of order 1, of order 2, of order 3, of order 4, up to isomorphism. (See exercises at the end of section 2.)

Homework 8, Friday, April 5.

(1*) Let $A, A', B \in \mathbb{R}^2$ be vectors such that $B \neq 0$, $A \perp B$, and $A' \perp B$. Prove that A and A' are proportional.

(2*) Let A, B, C be points in the plane such that $A \neq B$. Prove that there exists exactly one line passing through C and perpendicular to l_{AB}. (There are two statements here.)

(3) Let $\ell_{A'B'}$ be a line orthogonal to a line ℓ_{AB}, and $\ell_{C'D'}$ be a line orthogonal to a line ℓ_{CD}. Prove that $\ell_{AB}||\ell_{CD}$ if and only if $\ell_{A'B'}||\ell_{C'D'}$. (Hint: Use (1).)

(4) Let (A, B, C) be a triangle (as usual, this means nondegenerate).
   (a) If a and b are the perpendicular bisectors of sides $[B, C]$ and $[A, C]$, prove that a and b intersect at exactly one point. [Hint: You can use (3) and homework 3(2), for example.]
   (b) If l_A and l_B are two altitudes of this triangle, prove that l_A and l_B intersect at exactly one point.

(5) Page 60, exercises 3.1, 3.2, 3.4*, 3.6, 3.7*.

Topics: The nine-point theorem, isometry (different from the book), isometries of $\mathbb{R}^2$ fixing three points (statement), the Cauchy-Schwartz inequality, the triangle inequality for vectors, the triangle inequality for the distance in the plane (hw), translations are isometries, composition of isometries is an isometry, projection, linear functions, images of lines under linear maps, ...

For extra fun.

- Given two circles in the plane of the same radius, $\mathcal{C}$ and $\mathcal{C}'$, how many dilatations can you find that map $\mathcal{C}$ to $\mathcal{C}'$?
- Given two circles in the plane of different radii, $\mathcal{C}$ and $\mathcal{C}'$, how many dilatations can you find that map $\mathcal{C}$ to $\mathcal{C}'$?
- Let $D(\mathbb{R}^2)$ be the group of dilatations of $\mathbb{R}^2$ as before. Fix some point $X_0 \in \mathbb{R}^2$ and consider the map $\omega : D(\mathbb{R}^2) \to \mathbb{R}^2$ defined by $\omega(g) := g(X_0)$ for all $g \in D(\mathbb{R}^2)$. Prove that ω is surjective. Such a map ω is called an *orbit map* for the action of $D(\mathbb{R}^2)$ on $\mathbb{R}^2$.
- More generally, given any action of a group G on a set S, define an orbit map. If the action is *transitive*, prove that any orbit map for this action is surjective.
- For $D(\mathbb{R}^2)$ and X_0 as above, prove that the set $\{\alpha \in D(\mathbb{R}^2) \mid \alpha(X_0) = X_0\}$ is a subgroup of $D(\mathbb{R}^2)$. Such a subgroup is called the *stabilizer* of X_0 in $D(\mathbb{R}^2)$.
- More generally, state and prove a similar statement about stabilizers for an arbitrary action of a group G on a set S.
- Describe the elements of the stabilizer $\{\alpha \in D(\mathbb{R}^2) \mid \alpha(X_0) := X_0\}$ *explicitly.*
- For the orbit map $\omega : D(\mathbb{R}^2) \to \mathbb{R}^2$ as above, what is the preimage of the point X_0 under ω? More generally, what is the preimage of an arbitrary point $X \in \mathbb{R}^2$ under ω?
- Find examples, as many as you can, of groups acting transitively on sets and metric spaces. Explicitly describe stabilizers of points for those actions.

Homework 9, Friday, April 12.

(1) Page 73, exercise 3.11*, 3.13*. (In 3.13, E_1 and E_2 are *arbitrary* orthogonal vectors of unit length. If you use a parallelogram, first prove that it is indeed a parallelogram as defined at the beginning of the course. If you use the fact that X is the sum of its projections, then first prove this fact. It might be easier instead to write E_1 in coordinates, and then see what E_2 can be. Then write X in coordinates, and see.

As usual, there might be many ways to solve a problem. Practice creative thinking. Investigate what solutions can be there, and which ones are the best.)

(2*) Prove that the set of isometries of $\mathbb{R}^2$, denoted $Isom(\mathbb{R}^2)$, is a subgroup of $S(\mathbb{R}^2)$. (It is therefore a group of transformations as defined in class.)

(3) Page 83, exercise 4.2*.

Topics: Images of lines under isometries, isometries of $\mathbb{R}^2$ fixing three points (proof, better than in the book), examples of isometries: translation, central reflection, central reflection, is an isometry, reflection (about a line), ...

For extra fun.

- Pick your favorite metric space (X, d). Give examples of isometries of X. Describe the group of isometries of X, $Isom(X)$, as explicitly as possible.
- In the hyperbolic plane $\mathbb{H}^2$ (see Escher's picture above), describe a metric on $\mathbb{H}^2$ with respect to which all the angels are of the same size, that is, are *isometric* to each other. (You might need some *Riemannian geometry* to do that.) With respect to that metric on $\mathbb{H}^2$, what can you say about $Isom(\mathbb{H}^2)$? Are there interesting examples of isometries of $\mathbb{H}^2$?
- Prove that, for any linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(0) = 0$.

Exam 3, Friday, April 19.

Projects.

Topics: Reflections are isometries, glide reflections, classification of isometries, ...