Homework, topics, practice test, fun.
Below, “*” means “turn in”, “no *” means “do not turn in, but know how to solve”. If a homework is in yellow color, it is still at a preliminary stage and might be modified later, but feel free to start working on it. I will also include an incomplete list of topics. Your practice test consists of:

- homework (listed below),
- topics (listed below),
- your class notes, and
- the textbook material covered in class.

Use them all as a guide to prepare for exams. If any questions arise, please ask me. Coming to office hours is also a very good idea. The problems marked “for extra fun” are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration.

Homework 1. Due Thursday, January 25.

1(*) Let $A$ be any set. Prove that the power set $P(A)$ is a Boolean algebra. Give a complete proof.

2(*) Let $A$ be a set. The set of all functions $f : A \to \{0, 1\}$ is denoted $2^A$. Define a “reasonable” one-to-one correspondence between the power set $P(A)$ and $2^A$. Prove that it is indeed a one-to-one correspondence. Describe the operations in $2^A$ that correspond to the operations $\cup$, $\cap$, $'$ (complement) in $P(A)$. Describe the relation on $2^A$ that corresponds to the relation $\subseteq$ on $P(A)$. What are the top element and the bottom element of $2^A$? Can $2^A$ be viewed as a Boolean algebra? Why?

3) Inclusion, section 1.1, p.3: # 2, 3, 4, 6, 7, 8*.

4) Operations on sets, section 1.2, p.8: # 1, 2, 3, 7, 8*.

5*) Suppose that $(X, d)$ is a metric space. Prove that for any two (open) balls $B(a_1, r_1)$ and $B(a_2, r_2)$ in $X$, there exists a family $\mathcal{F}$ of (open) balls in $X$ such that

$$B(a_1, r_1) \cap B(a_2, r_2) = \bigcup_{B \in \mathcal{F}} B.$$ 

6*) Suppose that $(X, d)$ is a metric space. Prove that the family $\mathcal{T}$ associated with $(X, d)$, as was defined in class, is a topology on $X$. [Hint: Use (5).]

Topics: Set, element, subset, equality of sets, union, intersection, complement, properties of operations on sets, Cartesian product, relation, function, image of a set, preimage (= inverse image) of a set, metric, metric space, ball, topology, topological space, open set, from a metric space to a topological space, partial order, partially ordered set.

Homework 2. Due Thursday, February 1.

1*) Let $X$ be any set. Is $P(X)$ a topology on $X$? Justify the answer.


3) Functions, section 1.4, p.17: # 1, 2, 3*, 9*, 10*, 12, 16*.
(4) Relations, Cartesian product, section 1.5, p.20: # 1, 2*, 3*.
Relation between equivalence classes and partitions.

(5) Prove that it is not true that a least upper bound of a subset \( S \) of a partially ordered set \( L \) (more precisely, of \((L, \leq)\)) always exists.

(6) Prove that a least upper bound of a subset \( S \) of a partially ordered set \( L \) is unique.

**Topics:** Partially ordered set, total order (= linear order), chain (= totally ordered set= linearly ordered set), examples of partially ordered sets, power set \( P(X) \), lattice, Boolean algebra, upper bound for \( S \subseteq L \), least upper bound (= smallest upper bound), lower bound, greatest lower bound (= largest upper bound), injective (= one-to-one), surjective (= onto), bijection (= one-to-one correspondence), finite set, countably infinite set, countable set, countable union of countable sets, having the same cardinality for two sets, cardinality of \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \).

**For fun:** Pick a set and exhibit an equivalence relation on it. Describe the corresponding partition explicitly. Repeat.

**Homework 3. Due Thursday, February 8.**

(1) Countable sets, section 2.1, p. 26: # 1*, 2, 3*, 10*.

(2) Cardinal numbers, section 2.2, p. 31: # 1, 2, 3*.

(3*) How does the cardinality of the set of irrational numbers compare to the cardinality of \( \mathbb{R} \)? (Equal, strictly less, or strictly larger?) Give a full proof for the answer.

(4) Prove reflexivity for cardinality: if \( A \) is a set, then \( o(A) \leq o(A) \).

(5) Prove transitivity for cardinality: if \( A, B, C \) are sets such that \( o(A) \leq o(B) \) and \( o(B) \leq o(C) \), then \( o(A) \leq o(C) \).

**Topics:** Comparing the cardinalities of \( A \) and \( P(A) \), infinitely many cardinalities, Russel’s paradox, equivalence relation, equivalence class, partition (see homework 2), \( o(A) = o(B) \), \( o(A) \leq o(B) \), \( o(A) < o(B) \), antisymmetry for cardinality, totality (i.e. comparability) for cardinality, Zorn’s lemma.