Below, “∗” means “turn in”, “no ∗” means “do not turn in, but know how to solve”. If a homework is in yellow color, it is still at a preliminary stage and might be modified later, but feel free to start working on it. I will also include an incomplete list of topics. Your practice test consists of:

- homework (listed below),
- topics (listed below),
- your class notes, and
- the textbook material covered in class.

Use them all as a guide to prepare for exams. If any questions arise, please ask me. Coming to office hours is also a very good idea. The problems marked “for extra fun” are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration.

Homework 1. Due Thursday, January 25.

(1) Section 0, p.8: #1-12, 29-34.

(2*) For complex numbers $z = a + bi$ and $z' = a' + b'i$, define the product $zz'$, define the norm (magnitude, length) $|z|$, then prove that $|zz'| = |z||z'|$.

(3) Part 1, Section 3, p.34: #1-10.

(4*) Prove that the set of $m \times n$ matrices with real entries, $M_{m \times n}(\mathbb{R})$, under addition is a group.

(5*) Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Prove that the set of $n \times n$ matrices in $M_n(\mathbb{R})$ such that $\det M \neq 0$ is a group under matrix multiplication. (Feel free to use standard facts from linear algebra.)

(6) Part 1, Section 4, p.45: # 1*, 2*, 4*, 6, 10*, 11*, 12*, 13*, 14*, 15*, 16*, 17*, 18*. (For matrices, though not absolutely necessary, it might help to use the previous exercises and the subgroup test.)

Topics: Set, function, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, relation between $A$ and $B$, relation on $A$, equivalence relation, binary operation, binary structure, group, multiplicative and additive notations, abelian group, examples of groups and non-groups, matrices, uniqueness of identity, uniqueness of inverses.

Homework 2. Due Thursday, February 1.

(1*) Let $A$ be a nonempty collection of subgroups of a group $G$. Prove that the intersection $\bigcap_{H \in A} H$ is a subgroup of $G$.

(2) Part 1, Section 4, p.47: # 23*.

(3) Part 1, Section 5, p.55: # 13*, 52*, 54. ($H_S$ is called the centralizer of $S$ in $G$. $H_G$ is the center of $G$.)

Topics: Cancellation laws in groups, subgroup, subgroup test, cyclic group, integers mod $n$, $\mathbb{Z}_n$ (two definitions), (Euclidean) division algorithm (more than in the book), greatest common divisor (different from book), classification of cyclic groups (up to isomorphism), classification of subgroups of $\mathbb{Z}$, classification of subgroups of $\mathbb{Z}_n$, subgroups of cyclic groups, order of a group, order of an element, injective, surjective, bijective, homomorphism, isomorphic groups (like different languages), $\mathbb{R}_{2\pi}$, $\mathbb{R}_c$, $GL(n, \mathbb{R})$ (described in homework), equivalence classes, partition, Cayley graph (it is less confusing to always draw directed edges), subgroup generated by a set, permutation of a set.

For extra fun:

(1) Consider the motion of Rubik’s cube that first turns a side face 90° clockwise and then turns the top face 90° clockwise. This is an element of the group that I described in class. What is the order of this element?

(2) Is the group associated with Rubik’s cube abelian?

(3) Formally and rigorously define what “a motion” of Rubik’s cube is. We might also call it “a transformation”. Denote the set of all transformations by $T$. Define a binary operation on $T$ and show that with this operation $T$ becomes a group.

(4) What “should” be a good choice for “elementary transformations” of Rubik’s cube? How do they relate to all transformations?

(5) Consider the rotation of a face of Rubik’s cube 90° clockwise and the rotation of the same face 270° counterclockwise. “Should” they be treated as the same transformation or as different?

Homework 3. Due Thursday, February 8.

(1*) Show that the image of a subgroup $E \leq H$ under a group homomorphism $\varphi : H \to G$ is a subgroup of $G$.

(2*) Prove that the following statements are equivalent for any function $f : A \to B$.

(a) $f$ is a bijection.

(b) There exists a function $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$. (Here $id_A : A \to A$ denotes the identity function on $A$.)

(3) Prove that the order of the symmetric group $S_n$ is $n!$.

(4) Part 1, Section 7, p.72: # 7.

(5*) Draw the Cayley graphs for the indicated groups and generating sets: $(\mathbb{Z}, \{1\})$, $(\mathbb{Z}, \{2, 3\})$, $(\mathbb{Z}_8, \{1\})$, $(\mathbb{Z}_8, \{3\})$. (It is always better to use only directed edges.)