Below, “∗” means “turn in”, “no ∗” means “do not turn in, but know how to solve”. If a homework is in yellow color, it is still at a preliminary stage and might be modified later, but feel free to start working on it. I will also include an incomplete list of topics. Your practice test consists of:

- homework (listed below),
- topics (listed below),
- your class notes, and
- the textbook material covered in class.

Use them all as a guide to prepare for exams.

How to test whether you are prepared for the exams? Close your book and notes, and write the (whole) proof for a theorem or solution for a problem. If you get stuck, go back and read, then, again, start from the beginning and write the whole proof without looking in the book or notes.

If any questions arise, please ask me. Coming to office hours is also a very good idea, this is what they are for. Take initiative and come with questions.

The problems marked “for extra fun” are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration.

Homework 1. Due Thursday, January 25.

1. Section 0, p.8: #1-12, 29-34.
2. For complex numbers $z = a + bi$ and $z' = a' + b'i$, define the product $zz'$, define the norm (magnitude, length) $|z|$, then prove that $|zz'| = |z||z'|$.
3. Part 1, Section 3, p.34: # 1-10.
4. Prove that the set of $m \times n$ matrices with real entries, $M_{m \times n}(\mathbb{R})$, under addition is a group.
5. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Prove that the set of $n \times n$ matrices in $M_n(\mathbb{R})$ such that $detM \neq 0$ is a group under matrix multiplication. (Feel free to use standard facts from linear algebra.) This group is called the general linear group and is denoted $GL(n, \mathbb{R})$.
6. Part 1, Section 4, p.45: # 1*, 2*, 4*, 6, 10*, 11*, 12*, 13*, 14*, 15*, 16*, 17*, 18*. (For matrices, though not absolutely necessary, it might help to use the previous exercises and the subgroup test.)

Topics: Set, function, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, relation between $A$ and $B$, relation on $A$, equivalence relation, binary operation, binary structure, group, multiplicative and additive notations, abelian group, examples of groups and non-groups, matrices, uniqueness of identity, uniqueness of inverses.

Homework 2. Due Thursday, February 1.

1. Let $A$ be a nonempty collection of subgroups of a group $G$. Prove that the intersection $\bigcap_{H \in A} H$ is a subgroup of $G$. 

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(2) Part 1, Section 4, p.47: # 23*.
(3) Part 1, Section 5, p.55: # 13*, 52*, 54. (\(H_S\) is called the centralizer of \(S\) in \(G\). \(H_G\) is the center of \(G\).)

**Topics:** Cancellation laws in groups, subgroup, subgroup test, cyclic group, integers mod \(n\), \(\mathbb{Z}_n\) (two definitions), (Euclidean) division algorithm (more than in the book), greatest common divisor (different from book), relatively prime numbers (= mutually prime), subgroups of cyclic groups, classification of cyclic groups (up to isomorphism), order of a group, order of an element, orders of subgroups of \(\mathbb{Z}_n\), generators of a finite cyclic group, injective, surjective, bijective, homomorphism, isomorphism, isomorphic groups (like different languages), \(\mathbb{R}_{2\pi}, GL(n, \mathbb{R})\) (described in homework), permutation of a set, the symmetric group \(S_A, S_n\), equivalence classes, partition, orbit of a permutation, cycle.

**For extra fun:**

(1) Consider the motion of Rubik’s cube that first turns a side face 90° clockwise and then turns the top face 90° clockwise. This is an element of the group that I described in class. What is the order of this element?
(2) Is the group associated with Rubik’s cube abelian?
(3) Formally and rigorously define what “a motion” of Rubik’s cube is. We might also call it “a transformation”. Denote the set of all transformations by \(T\). Define a binary operation on \(T\) and show that with this operation \(T\) becomes a group.
(4) What “should” be a good choice for “elementary transformations” of Rubik’s cube? How do they relate to all transformations?
(5) Consider the rotation of a face of Rubik’s cube 90° clockwise and the rotation of the same face 270° counterclockwise. “Should” they be treated as the same transformation or as different?

**Homework 3. Due Thursday, February 8.**

(1*) Prove that any homomorphism \(\varphi : G \to H\) sends identities to identities, and inverses to inverses, that is, \(\varphi(1_G) = 1_H\), and for any \(a \in G\), \(\varphi(a^{-1}) = \varphi(a)^{-1}\).
(2*) Prove that the image of a subgroup \(A \leq G\) under a group homomorphism \(\varphi : G \to H\) (denoted \(\varphi(A)\) or \(\varphi[A]\)) is a subgroup of \(H\).
(3*) Prove that the following statements are equivalent for any function \(f : A \to B\).
(a) \(f\) is a bijection.
(b) There exists a function \(g : B \to A\) such that \(g \circ f = id_A\) and \(f \circ g = id_B\). (Here \(id_A : A \to A\) denotes the identity function of \(A\).)
(4) Prove that the following statements are equivalent for any group homomorphism \(f : G \to H\).
(a) \(f\) is an isomorphism.
(b) There exists a homomorphism \(g : B \to A\) such that \(g \circ f = id_A\) and \(f \circ g = id_B\).
(c) There exists an isomorphism \(g : B \to A\) such that \(g \circ f = id_A\) and \(f \circ g = id_B\).
(5) Prove that the order of the symmetric group \(S_n\) is \(n!\).
(6) Part 2, Section 8, p.83: # 1, 44*, 45*. 
(7*) Draw the Cayley graphs for the indicated groups and generating sets: \((\mathbb{Z}, \{1\}), (\mathbb{Z}, \{2, 3\}), (\mathbb{Z}_8, \{1\}), (\mathbb{Z}_8, \{3\})\). (It is better to use only directed edges, and label the edges as well.)

**Topics**: Orbit of a permutation, cycle (= cyclic permutation), disjoint cycles commute, permutations as products of cycles, subgroup generated by a set, Cayley graph (precise definition that is missing in the book; it is better to draw directed edges and also label the edges), transposition, permutations as products of transpositions, even and odd permutations, the alternating group \(A_n\), homomorphism \(S_n \to \mathbb{Z}_2\), image of a subgroup under homomorphism (see homework), preimage of a subgroup, image of a homomorphism, kernel.

**Homework 4. Due Thursday, February 15.**

1. Prove that each \(f \in S_n\) can be uniquely represented as a product of disjoint cycles. By “uniquely” here we mean “uniquely up to permutation of cycles, up to permutation of the numbers occurring in each cycle, and up to removing or adding cycles of length 1”. Note that this means that we need to prove both the existence of such a decomposition and its uniqueness.

2. Prove that disjoint cycles in \(S_n\) commute. I.e. if \((a_1 \ldots a_p) \in S_n\), \((b_1 \ldots b_q) \in S_n\), and \(\{a_1 \ldots a_p\} \cap \{b_1 \ldots b_q\} = \emptyset\), then \((a_1 \ldots a_p)(b_1 \ldots b_q) = (b_1 \ldots b_q)(a_1 \ldots a_p)\).

3. Let \(G\) be a group and \(A, B, C\) be subsets of \(G\) (not necessarily subgroups). Denote \(AB := \{ab \mid a \in A, b \in B\}\). Prove the associativity of this set operation: \((AB)C = A(BC)\). Therefore, we can unambiguously write \(ABC\) instead of \((AB)C\) or \(A(BC)\).

4. Suppose \(G\) is a group and \(H \leq G\). Show that each left coset \(xH\) equals a “product” of subsets as above. What are those subsets?

5. Is \(S_n\) abelian? Justify the answer.

6. Part 2, Section 9, p.94 # 1, 2*, 3, 4*, 6*, 10*, 19*, 36*.

7. Draw the Cayley graphs for the indicated groups and generating sets. (Again, use only directed edges, and do not forget to label the edges.)
   - (a) \((\mathbb{Z} \times \mathbb{Z}, \{a, b\})\), where \(a := (1, 0)\) and \(b := (0, 1)\).
   - (b) \((\mathbb{Z} \times \mathbb{Z}_3, \{a, b\})\), where \(a := (1, [0])\) and \(b := (0, [1])\).
   - (c) \((\mathbb{Z}_2 \times \mathbb{Z}_2, \{a, b\})\), where \(a := ([1], [0])\) and \(b := ([0], [1])\). (This group is called the **Klein 4-group**.)

**Topics**: Direct product of \(n\) groups (= Cartesian product), a group of permutations, Cayley’s theorem, left and right cosets for \(H \leq G\), representing cosets as equivalence classes, index, cosets are copies of the subgroup, the theorem of Lagrange, normal subgroup (two definitions), factor group (= quotient group).

**Homework. To know before Exam 1.**

1. Fix a set \(A\). Given an equivalence relation \(\sim\) on \(A\), define a partition of \(A\) that corresponds to \(\sim\). Prove that this is indeed a partition.

2. Fix a set \(A\). Given a partition \(\{A_i \mid i \in I\}\) of \(A\), define an equivalence relation on \(A\) that corresponds to the partition. Prove that it is indeed an equivalence relation.

3. Prove that the map \(\varphi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}\) given by \(\varphi(a) := [a]\) is a homomorphism.

4. Show that the composition of two homomorphisms (of groups) is a homomorphism. Then show that the composition of two isomorphisms (of groups) is an isomorphism.

5. Part II, Section 10, p. 101: # 1-5, 12, 14, 26, 27.
Topics: The quotient homomorphism $\gamma : G \to G/H$, kernel is normal, the first isomorphism theorem with proof (better than in the book).

For extra fun:

- Prove that for each set $A$, there exists a bijection between the set of all equivalence relations on $A$ and the set of all partitions of $A$.

Below are just some randomly chosen topics that you might consider for your project, they are intended as a guide only. Do your own research, suggest your own topic. Find what you like. Sky is the limit. Discuss your possible topic with me in advance.

1. Fundamental groups of topological spaces: how groups arise in topology.
2. Group actions on various geometric objects, their properties.
3. Graphs, groups, Cayley graphs, how to visualize groups geometrically.
4. Free groups, their Cayley graphs, groups acting on trees.
5. Patterns, tessellations, and groups acting on them.
6. Vector spaces, matrices, how they are related to groups and fields.
7. How groups are used in physics.
8. How groups are related to music.
10. Part II, Section 11, p. 104. Finitely generated abelian groups, their classification.
11. Symmetries of any algebraic objects form a group. For example, all automorphisms of a given group.
12. Semidirect products of groups, examples.
13. , (14), (15), .......................................................

Homework 5. Due Thursday, March 1.

1. Memorize the statements of the three isomorphism theorems for groups. Be able to prove the first isomorphism theorem.
2* If $\varphi : G \to H$ is a homomorphism, prove that $\varphi$ is injective if and only if $\text{Ker}\varphi = \{1\}$.
3* Prove that the alternating group $A_n$ is normal in $S_n$.
4* Let $\text{Aut}(G)$ be the set of all automorphisms of a group $G$, as defined on page 141. Prove that composition,

$$ \circ : \text{Aut}(G) \times \text{Aut}(G) \to \text{Aut}(G), $$

is a well-defined operation on $\text{Aut}(G)$, and that $(\text{Aut}(G), \circ)$ is a group.
5* Let $\text{Inn}(G)$ be the set of all inner automorphisms of a group $G$, as defined on page 141. Prove that $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$.

Topics: When $HN \leq G$, the statements of the second and third isomorphism theorems, $S_n$-action on $\{1, \ldots, n\}$, automorphism of a group, $\text{Aut}(G)$-action on $G$, group action on a set, $G$-set, $G$-acton as a homomorphism, orbit of an action, stabilizer (= isotropy group), orbit-stabilizer relation, Burnside’s lemma (theorem, formula), Sylow theorems.

For extra fun:

1. Prove the second and third isomorphism theorems for groups.
(2) Fix a positive integer $n$. Consider an action of $\mathbb{Z}$ on $\mathbb{R}^2$ such that $1 \cdot x$ is the result of rotating $x \in \mathbb{R}^2$ by the angle of $2\pi/n$ radian. Show that there is only one such action. Describe it explicitly. Describe all the orbits of this action.

(3) What if $\mathbb{Z}$ is replaced with $\mathbb{R}$ in the above problem? (Speaking of the term “orbit”.)

(4) Define group action on a graph. Prove that each group acts on each of its Cayley graphs.

**Homework 6. Due Thursday, March 8.**

(1) Isomorphism theorems: Part IV, section 34, page 310: # 1, 2, 7*, 8*.

(2*) Suppose that $\sigma \in S_n$ is represented as a product of disjoint cycles:

$$\sigma = (a_1 \ldots a_{m_1})(a_{m_1+1} \ldots a_{m_2}) \ldots (a_{m_k-1+1} \ldots a_{m_k}).$$

Let $\langle \sigma \rangle := \{\sigma^i \mid i \in \mathbb{Z}\}$ be the cyclic subgroup of $S_n$ generated by $\sigma$. Consider the action of $S_n$ on $\{1, \ldots, n\}$. Explicitly describe all the orbits of this action. The above action by $S_n$ can be restricted to an action of $\langle \sigma \rangle$ on $\{1, \ldots, n\}$. Explicitly describe all the orbits of this restricted action as well.

(3*) Use Burnside’s lemma to compute the number of rotationally distinct colorings of the cube using $n$ colors.

For extra fun:

(1) If a soul is open to inspiration, perfectly mundane things can inspire. On the left is a picture of the waste basket in my office. What can you say about its group of symmetries? (You might need to come to my office hours to solve this problem.)

(2) **Cherish your very own group.** Pick any object you like. Describe what kind of a structure it has. Then describe what you would mean by a symmetry of this object that preserves its structure. Let $G$ be the set of all such symmetries. Convince yourself that $G$ is a group. What can you say about $G$? What properties does it have?

(3) Repeat the above until slightly tired. It is a great exercise for mathematical inspiration.

(4) Make sure to take rest and get enough sleep. Come to talk to me if any questions arise.

(5) What groups admit “reasonable” actions on the plane $\mathbb{R}^2$? On the space $\mathbb{R}^n$? On the sphere $S^2$?