Below, “∗” means “turn in”, “no ∗” means “do not turn in, but know how to solve”. If a text is in yellow color, the homework is still at a preliminary stage and might be modified later, but feel free to start working on it. The problems marked “for extra fun” are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration. I will also include an incomplete list of topics.

Homework 1. Due Friday, September 7.

(1*) Prove that every topology on $X$ is a basis for itself. (There are two statements here.)

(2) Part 13, Basis for a topology, p. 83: # 1, 3, 4*, 5*, 8a*.

(3) Part 16, The subspace topology, p. 91: # 1*, 3.

Topics: A metric, a topology, a topological space, a basis for a topology, open set, the transition metric $\sim$ topology, the topology generated by a basis (two definitions), comparing topologies, comparing bases, a subbasis, the transition subbasis $\sim$ basis $\sim$ topology, a sufficient criterion for a basis of $T$, the subspace topology, a basis for the subspace topology, closed set, the topology of disjoint union, ...

Homework 2. Due Friday, September 14.

(1*) Consider a disjoint union of topological spaces, $Z := \bigsqcup_{i \in I} Z_i$, with the disjoint union topology. Prove that a subset $U \subseteq Z$ is open in $Z$ if and only if for any $i \in I$, $U \cap Z_i$ is open in $Z$. Prove that a subset $V \subseteq Z$ is closed in $Z$ if and only if for any $i \in I$, $V \cap Z_i$ is closed in $Z$.

(2*) Suppose $Z := \bigsqcup_{i \in I} Z_i$ is a disjoint union as in (1) and $f : Z \to A$ is a surjective function. Put the quotient topology on $A$. Prove that a subset $U \subseteq A$ is open in $A$ if and only if for all $i \in I$, $f^{-1}(U) \cap Z_i$ is open in $Z$. Prove that a subset $V \subseteq A$ is closed in $A$ if and only if for all $i \in I$, $f^{-1}(V) \cap Z_i$ is closed in $Z$.

(3*) What does this say about the open/closed subsets of a cell complex? State and justify.

(4*) Given a topological space $X$ and a surjective function $f : X \to A$, prove that the quotient topology on $A$ is the largest (= finest, strongest) topology on $A$ with respect to which $f$ is continuous.

(5*) Give two definitions of the projective plane and prove that they give homeomorphic topological spaces.

(6) Let $A$ be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Give several (at least four) equivalent definitions of continuity of $f$. Prove that they are indeed equivalent.

Topics: 2-sphere, projective plane, the quotient topology (without using continuity), building new topological spaces, disk $D^n$, gluing and equivalence relations, an informal definition of a cell complex, homeomorphism, saturated sets, continuous function, continuity in terms of bases, other definitions of homeomorphism, continuity of constant function, of composition, of restriction, standard simplex $\Delta^n$, an informal definition of a simplicial complex, product topology on $X \times Y$, continuity of projections, continuity of functions into product $X \times Y$.

For extra fun.

- Take the standard sphere $S^2$ (with the subspace topology) and glue each point $a \in S^2$ to its antipodal point $-a$. What “reasonable” topology can you put on the result of this gluing? The resulting topological space is called the projective plane.
projective plane be viewed as a cell complex? What is the minimal number of cells in a cell complex like this? Can the projective plane be viewed as a simplicial complex? What is the minimal number of simplices needed? What other interesting properties of the projective plane can you discover?

- Define the 3-dimensional analog of the projective plane (called the 3-dimensional projective space) in two ways and investigate what properties it has.
- Let $B^n$ denote the $n$-dimensional (open) ball, with the standard topology.
  - Prove that $B^0$ and $B^1$ are not homeomorphic.
  - Prove that $B^1$ and $B^2$ are not homeomorphic.
  - Prove that $B^2$ and $B^3$ are not homeomorphic.

**Homework 3. Due Friday, September 21.**

1. Let $X_i$ be a topological space for each $i \in I$. Prove that the product topology on $\prod_i X_i$ is the smallest (=coarsest, weakest) topology with respect to which all the projections $\pi_j : \prod_i X_i \to X_j$ are continuous.

2. Prove the following **universal property** for the cartesian product of sets: given a set $Y$, an indexed family of sets $\{X_i \mid i \in I\}$, and a family of functions $\{\varphi_i : Y \to X_i \mid i \in I\}$, then there exists a unique function $\varphi : Y \to \prod_i X_i$ that makes the following diagram commute for each $j \in I$, i.e. $\pi_j \circ \varphi = \varphi_j$.

3. Prove the following **universal property** for the cartesian product of topological spaces: given a topological space $Y$, an indexed family of topological spaces $\{X_i \mid i \in I\}$, and a family of continuous functions $\{\varphi_i : Y \to X_i \mid i \in I\}$, then there exists a unique continuous function $\varphi : Y \to \prod_i X_i$ that makes the above diagram commute for each $j \in I$, i.e. $\pi_j \circ \varphi = \varphi_j$.

4. Let $Y$ be a topological space, $\prod_i X_i$ be a product of topological spaces, and $f : Y \to \prod_i X_i$ be any function. Prove the following statement **in two ways**: $f$ is continuous if and only if each of its coordinate functions, $f_i$, is continuous. One way: directly.
   Another way: using the universal property.

(5) Part 18, Continuous functions, p. 111: # 1*(prove the “if and only if” statement), 3, 4*, 5, 11*, 12*.

(6) Part 19, Product topology, p. 118: # 1, 2, 3, 4, 10*.

**Topics:** The general cartesian product, product topology (=Tychonoff topology) in terms of subbasis and in terms of basis, comparison of box topology with product topology, continuity of functions into arbitrary products (see hw), Hausdorff topology, characterizations of continuity in terms of neighborhoods, in terms of closed sets, the closure of a subset $A \subseteq X$, the characterization of closed sets in terms of closure, a characterization of closure in terms of neighborhoods, characterization of continuity in terms of closure, interior, limit point ...

**For extra fun.**

- Suppose $\{X_i \mid i \in I\}$ is an indexed family of sets, and each $X_i$ is non-empty. Is $\prod_i X_i$ empty or not? For example, is $\prod_{i \in \mathbb{N}} \mathbb{R}$ empty or not? How can one be absolutely sure about the answer?
• Let $X_i$ be a topological space for each $i \in I$. Prove that the disjoint union topology on $\bigsqcup_i X_i$ is the largest (=finest, strongest) topology with respect to which all the inclusions $\iota_j : X_j \to \bigsqcup_i X_i$ are continuous.

• Look at the diagram

\[
\begin{array}{c}
Y \\
\varphi_j \downarrow \varphi \\
X_j \rightarrow \bigsqcup_i X_i
\end{array}
\]

Formulate the universal property for disjoint unions of sets, and then for disjoint unions of topological spaces. Prove these universal properties.

• Prove in two ways that a function $f : \bigsqcup_i X_i \to Y$ is continuous if and only if the restriction of $f$ to each $X_i$ is continuous.

Homework 4. Due Friday, September 28.

1. Consider a disjoint union of topological spaces, $Z := \bigsqcup_{i \in I} Z_i$, with the disjoint union topology, and suppose $W \subseteq Z_{i_0}$ for some $i_0 \in I$. Prove that $W$ is open in $Z$ if and only if $W$ is open in $Z_{i_0}$. Use this to give an equivalent restatement of problems (1), (2) and (3) from homework 2.

2. Part 17, Closed sets and limit points, p. 100: # 2 (see also Lemma 16.2), 3, 4, 6*, 7, 8, 9, 11, 13*.

3. Prove that each norm on a vector space indeed induces a metric.

4* Denote $\ell^1 := \ell^1(\mathbb{N}, \mathbb{R})$, $\ell^2 := \ell^2(\mathbb{N}, \mathbb{R})$. Put the $\ell^1$-topology on $\ell^1$ and the $\ell^2$-topology on $\ell^2$. Is the inclusion $\ell^1 \hookrightarrow \ell^2$ continuous? Justify the answer.

5* Put two topologies on $\ell^1$: the $\ell^1$-topology and the subspace topology induced from the $\ell^2$-topology on $\ell^2$. Do these topologies on $\ell^1$ coincide? Is any one smaller than the other? Justify the answers.

6. A map $f : X \to Y$ between topological spaces is called open if the image of any open set in $X$ is open in $Y$. Is the projection $\pi_j : \prod_{i \in I} X_i \to X_j$ open? Is the inclusion $\ell^1 \hookrightarrow \ell^2$ open? Justify the answers.

7. Part 20, The metric topology, p. 126: # 1, 3*.

8* Prove that each of the norms $(\ell^1, \ell^2, \ell^\infty)$ that we defined on $\mathbb{R}^n$ induces the standard topology on $\mathbb{R}^n$.

Topics: General definition of a norm, norms on $\mathbb{R}^n$ (without proof), metrics on $\mathbb{R}^n$, $\ell^1$, $\ell^2$, $\ell^\infty$, harmonic series, metric induced by a norm, metric topology, metrizable space, continuity of functions between metric spaces, relations between metric topologies, sequence, convergence of a sequence, closure in terms of sequences, continuity in terms of sequences, first-countable spaces, second-countable spaces, uniform convergence, uniform limit of continuous functions, ...

For extra fun.

• Prove that the $\ell^2$-norm on $\mathbb{R}^n$ and on $\ell^2$ is indeed a norm. Therefore it indeed gives rise to a metric. (See exercises 9 and 10, p. 128.) In this class, we will assume this fact without proof.

• Prove that the $\ell^\infty$-norm on $\mathbb{R}^n$ and on $\ell^\infty$ is indeed a norm.

• For what $p \in [0, \infty]$ is the $\ell^p$-norm on $\mathbb{R}^n$ and on $\ell^p$ indeed a norm?

Homework 5. Due Friday, October 5.

1. Part 21, The metric topology (continued), p. 133: # 1, 6, 8*.
(2) Part 23, Connected spaces, p. 152: # 1, 2, 5*.
(3*) Let \(X\) be a topological space, \(Y\) be a set, and \(f : X \to Y\) be a surjective function such that for each \(y \in Y\), the subspace \(f^{-1}(y)\) of \(X\) is connected. Put the quotient topology on \(Y\). Prove that \(X\) is connected if and only if \(Y\) is connected.
(4) Prove that \(\mathbb{R}^n\) is path-connected.
(5) Part 24, Connected subspaces of the real line, p. 157: # 1, 10*.

**Topics:** Connected space, path-connected space, convex subsets of \(\mathbb{R}\), intermediate value theorem for \(X \to \mathbb{R}\), continuous images of connected spaces, unions of connected spaces, connected component, closure of a connected subset, components are connected and closed, path-component, locally connected space, locally path-connected space, relation between connected components and path components, open cover (not covering), compact space ...

**For extra fun.**
- Look up the definition of a net in a topological space. Define convergence of a net.
- Prove that continuity of a function can be equivalently restated in terms of nets.
- Let \(A\) be a subset of a topological space \(X\). Prove that the closure of \(A\) in \(X\) can be equivalently defined in terms of nets.
- Prove that a cell complex is connected if and only if it is path connected.

**Homework. To know before the midterm exam, October 12.**
(2) Prove that \([0, 1]\) is compact. (See Corollary 27.2.) Explain why this does not follow from Theorem 27.3.
(3) Part 26, Compact spaces, p. 170: # 1, 3, 4, 5, 6.
(4) Let \(Y\) be a topological space and \(X \subseteq Y\) with the subspace topology. Consider the definition of compactness for \(X\) given in class, using covers of \(X\) by subsets of \(X\) that are open in \(X\). Give another definition of compactness for \(X\) using covers of \(X\) by subsets of \(Y\) that are open in \(Y\). Prove that the two definitions are equivalent.
(5) Part 27, Compact subspaces of the real line, p. 177: # 2.

**Topics:** Compact spaces, examples, subspaces and compactness, continuity and compactness, compactness of finite products, compact subspaces of \(\mathbb{R}^n\), ...