Homework 1. Due Friday, September 7.

(1) Prove Theorem 3.1, p. 15, without looking at the proof in the book. There might be many correct proofs one can present for a given statement. Try to find a shortest proof.

(2) Prove Theorem 3.2, p. 16, without looking at the proof in the book.

(3) Prove Theorem 3.5, p. 17, without looking at the proof in the book.

(4) Section 1.3, page 19: # 3.4*.

Part (v) of Theorem 3.2 cannot be deduced from the axioms of a field as they are stated in the book. First, state how the axioms of a field should be modified to be able to prove (v). Then give a detailed proof of (v) and (vii).

(5) Section 1.3, page 19: # 3.1, 3.5*, 3.6*, 3.7*.

(6*) A metric, or a distance function, on a set $X$ is a function $d : X \times X \to [0, \infty)$ such that

(a) $\forall x, y \in X \ d(x, y) = 0 \iff x = y$,
(b) $\forall x, y \in X \ d(x, y) = d(y, x)$,
(c) $\forall x, y, z \in X \ d(x, z) \leq d(x, y) + d(y, z)$.

Define the function $dist : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by the formula $dist(a, b) := |b - a|$ for $(a, b) \in \mathbb{R} \times \mathbb{R}$. Write a complete proof that this function $dist$ is a metric on $\mathbb{R}$. (Use only the axioms of the ordered field $\mathbb{R}$ and the properties that have been deduced from those axioms.)

Topics: Set, equality of sets, equivalence relation, natural numbers $\mathbb{N}$, the constructive definition of $\mathbb{Q}$, the constructive definition of the operations on $\mathbb{Q}$, an informal definition of $\mathbb{R}$, field, ordered field, $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields (without proof), uniqueness of inverses, properties of an ordered field, the absolute value, the metric on $\mathbb{R}$, maximum, minimum, upper bound (must be a number), lower bound, bounded above (correct definition), bounded below, sup, inf, the Archimedian property, the completeness axiom, denseness of $\mathbb{Q}$ in $\mathbb{R}$, ...

For extra fun:

(1) Prove that any two in the sequence of sets

$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$

are distinct. (Meaning they are not equal to each other as sets.) How can one formally and explicitly define the natural numbers?
(2) Prove that the operation $+$ and $\cdot$ that we defined on $\mathbb{Q}$ in class are indeed well-defined operations.
(3) Prove that $(\mathbb{Q}, +, \cdot)$ satisfy all the axioms of a field.
(4) Define an order relation $\leq$ on $\mathbb{Q}$ and prove that $(\mathbb{Q}, +, \cdot, \leq)$ satisfy all the axioms of an ordered field.
(5) Construct a set $A$ and define a notion of (unique) successor for each element of $A$ in such a way that $A$ satisfies all the Peano axioms except for the last one.

**Homework 2. Due Friday, September 14.**

(1) Section 1.4, p. 26: # 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7*, 4.9*, 4.11, 4.13, 4.14*, 4.16.
(2*) Prove that $\text{min} \ S$ and $\text{max} \ S$ exist (as numbers) for any nonempty finite subset $S$ in $\mathbb{R}$.
(3) Section 1.5, p. 30: # 5.4*, 5.5, 5.6*.

**Topics:** $\infty$ and $-\infty$, $\bar{\mathbb{R}}$, what it means for sup to exist, the meaning of sup $S = s_0$, sup $S = \infty$, countability, function, sequence, convergence to a real number, convergence to $\pm \infty$ (not divergence), what it means for lim to exist, and to not exist, sequences convergent in $\mathbb{R}$ are bounded, limit theorems: multiplication by a scalar, sum, product, inverse, ratio, ...

**Homework 3. Due Friday, September 21.**

(1) Read and understand the proof of Theorem 9.7 (basic examples of limits) on p. 48.
(2) Section 2.8, p.44: # 8.3, 8.4, 8.5*, 8.6, 8.7, 8.9*, 8.10* (assume $(s_n)$ is convergent in $\mathbb{R}$).

**Topics:** Limits in $\mathbb{R} \cup \{\infty\}$ for product and inverse, increasing/decreasing sequence, monotone, bounded monotone sequences converge (in $\mathbb{R}$), example: decimal fractions indeed represent real numbers, monotone sequences converge in $\mathbb{R}$, lim inf, lim sup, their existence, relation between lim inf, lim sup and lim (see hw), Cauchy sequence, convergent (in $\mathbb{R}$) $\Rightarrow$ Cauchy, Cauchy $\Rightarrow$ bounded, convergent (in $\mathbb{R}$) $\iff$ Cauchy, ...

**Homework 4. Due Friday, September 28.**

(1) Let $(s_n)$ be a sequence. Prove that if $\lim s_n$ exists in $\mathbb{R}$, then $\lim \inf s_n = \lim s_n = \lim \sup s_n$. (See Theorem 10.7(a), p. 61.)
(2) Let $(s_n)$ be a sequence. Prove that if $\lim \inf s_n = \lim \sup s_n$ (in $\mathbb{R}$), then $\lim s_n$ exists in $\mathbb{R}$ and $\lim \inf s_n = \lim s_n = \lim \sup s_n$. (See Theorem 10.7(b), p. 61.)
(3) Section 2.10, p.64: # 10.1, 10.2, 10.3*, 10.5, 10.7.
(4) Section 2.11, p.76: # 11.1, 11.2*, 11.3*, 11.8, 11.11*.
(5) Section 2.12, p.82: # 12.1*, 12.4*, 12.5.

**Topics:** Subsequences, convergence of sequences and subsequences, subsequential limits, the existence of monotone subsequences, the Bolzano-Weierstrass theorem for $\mathbb{R}$, metric space, $\mathbb{R}^k$, metrics in $\mathbb{R}^k$ ($d_2$, $d_1$, $d_\infty$, without proof), convergence in a metric space, Cauchy sequence in a metric space, complete metric space, Cauchy sequences in a metric space, complete coordinates in $\mathbb{R}^k$, convergence of sequences in $\mathbb{R}^k$, Cauchy sequences in $\mathbb{R}^k$, $\mathbb{R}^k$ is complete, Bolzano-Weierstrass in $\mathbb{R}^k$, open (sub)sets in a metric space $(S, d)$, closed (sub)sets in a metric space, ...

**Homework. To know before Exam 1.**

(1) Section 2.13, Some topological concepts in metric spaces, p.93: # 13.1, 13.2, 13.4, 13.5, 13.7, 13.13. (In all problems, change the notations to $d_2$, $d_1$, $d_\infty$ as defined in class; do not use the textbook’s notation for these metrics.)
(2) Prove Proposition 13.9(b). (Closed subsets in terms of sequences.)
Topics: Properties of open sets (hw), the intersection of a decreasing sequence of closed bounded nonempty subsets in $\mathbb{R}^k$, Cantor set, open cover of a subset in a metric space, compact subset of a metric space, the statement of the Heine-Borel theorem, series, what it means to converge/diverge, convergence to $\pm \infty$ (not divergence to $\pm \infty$), Cauchy criterion for series, geometric series.

For extra fun:

(1) A subset $E$ of a metric space $X$ (or more generally, a topological space $X$) is called sequentially compact if any sequence has a subsequence that converges in $E$. Restate the Bolzano-Weierstrass theorem using this notion.

(2) What is the relation between sequential compactness and compactness as defined in class?

(3) Restate the Bolzano-Weierstrass theorem using compactness and prove it.

(4) Prove that if $X$ is a metric space, then it is compact if and only if it is sequentially compact.

Homework 5. Due Friday, October 12.


(2) Know the proof of Theorem 12.2, p. 79-80.

Topics: Absolute convergence, comparison test, limit of the terms of a converging series, root test, $\sum 1/n^p$, harmonic series, ratio test, the statement of the integral test, the statement of the alternating series test, ...

Homework 6. Due Friday, October 19.

(1) Section 3.17, Continuous functions, p. 130: # 17.1 (domain here means the maximal subset of $\mathbb{R}$ on which the formula for the function makes sense), 17.2, 17.3, 17.4*, 17.5*, 17.6, 17.8* (give two proofs for 17.8(c): one using Example 5, the other the direct proof), 17.9*, 17.10*.

(2) Section 3.18, Properties of continuous functions, p. 138: # 18.8, 18.9.

Topics: Continuity at a point of a function $S \to \mathbb{R}$ (two definitions and their equivalence), continuous function, continuity for $f : (X, d_X) \to (Y, d_Y)$ (two definitions and their equivalence), continuity of $|f|, kf, f+g, fg, f/g, g \circ f$, maximum/minimum of a function on $[a, b]$, intermediate value theorem for $[a, b] \to \mathbb{R}$, uniform continuity for $f : S \to \mathbb{R}$ and for $f : (X, d_X) \to (Y, d_Y)$, characterization of continuity for $f : (X, d_X) \to (Y, d_Y)$ in terms of open sets, continuity on $[a, b] \Rightarrow$ uniform continuity, $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a} f(x) = -\infty$, $\lim_{x \to a} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = L$, $\lim_{x \to \infty} f(x) = L$, $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to a^+}$, $\lim_{x \to a^-}$, $\lim_{x \to a^-}$, ...

Homework 7. Due Friday, October 26.


Topics: Limits of $f+g$, $fg$, $f/g$, image of compact metric space under a continuous function, continuous on compact metric space $\Rightarrow$ uniformly continuous, maximum and minimum
for functions \( f : (X, d_X) \to (Y, d_Y) \) on compact \( X \), disconnected metric space, disconnected set in a metric space, connected set, continuity of \( f : X \to Y \) and \( f' : X \to f(X) \), image of a connected metric spaces under a continuous map, characterization of connected subsets of \( \mathbb{R} \), the more general intermediate value theorem for \( f : (X, d_X) \to \mathbb{R} \), ...

Some randomly chosen topics that you might consider for your project. These are intended as a guide only. Suggest your own topic. It should be related to the course in some way.

1. Countable sets, uncountable sets, cardinality of sets.
2. A rigorous definition of the set of real numbers. There are (at least) three approaches to it: using Dedekind cuts, using Cauchy sequences, using Peano axioms. Use one or several approaches.
3. A rigorous definition of its operations + and \( \cdot \) on \( \mathbb{R} \). A proof that \((\mathbb{R}, +, \cdot)\) is an ordered field.
4. After \( \mathbb{R} \) is defined, prove the completeness axiom from the definition, rather than assuming it.
5. Various notions of distance in various spaces.
6. Shapes in space, manifolds, how to glue a manifold out of pieces.
7. Lorentzian geometry, space and time.
8. Riemannian structure, Riemannian manifolds, tangent vectors, the intrinsic metric (= path metric) on a manifold.
10. Differential forms, how to generalize Green’s theorem, Stokes’ theorem, Gauss’ theorem (= the divergence theorem).
11. Uses of real analysis in physics, other subjects.
12. Various kinds of integrals. Why do we need different kinds? Their uses and applications.
13. An introduction to the Riemann hypothesis. Maybe a proof as well?
14. Black holes, Big Bang, curvature of space, etc.
15. Volume, measure of a set.
16. What notion of dimension can be defined for a metric space?
17. Cantor set, other interesting subsets of \( \mathbb{R} \) and \( \mathbb{R}^k \), other interesting metric spaces.
18. (18), (19), .......................................................

Homework 8. Due Friday, November 2.


2. Let \( S \) be any subset of \( \mathbb{R} \). Show that \( C(S) \) is connected. (cf. Exercise 22.9)

3. Show that \( \limsup kx_n = k \limsup x_n \) if \( (x_n) \) is bounded. (Exercise 12.6.)


Topics: Path-connected metric space, convex subsets of \( \mathbb{R}^k \), spaces of functions, the space \( C(S) \), power series, radius of convergence, use of \( \lim |a_{n+1}/a_n| \), interval of convergence, pointwise convergence and uniform convergence of a sequence of functions \( f_n : S \to \mathbb{R} \), uniform convergence and integrals, uniform limits of continuous functions, ...

Homework 9. Due Friday, November 9.


**Topics:** Uniformly Cauchy sequence \((f_n)\), uniformly Cauchy \(\Rightarrow\) uniformly convergent (the converse is hw), uniformly convergent series, the limit of uniformly convergent series of continuous functions is continuous, the uniform Cauchy criterion for series of functions \((\Leftrightarrow\) series is uniformly Cauchy), Weierstrass M-test, examples, uniform convergence of power series on each \([-R_1,R_1] \subseteq (-R,R)\), the limit of \(\sum a_n x^n\) on \((-R,R)\) is continuous, integration of power series, derivative, derivative of \(x^n\) for \(n \in \mathbb{N}\), differentiation of power series, ...

**For extra fun:**

1. Define a function \(f: \mathbb{R} \to \mathbb{R}\) that is continuous at each point in \(\mathbb{R}\) but not differentiable at \textit{any} point in \(\mathbb{R}\). Prove that it indeed has these properties. (Cf. Example 3 on p. 204 and Exercise 25.11, p. 207.)
2. Define a continuous function \(f: [0,1] \to [0,1] \times [0,1] \subseteq \mathbb{R}^2\) that is surjective, that is, each \((x,y) \in [0,1] \times [0,1]\) is a value of \(f\). Prove that it is indeed continuous and surjective.

**Homework. To know before Exam 2.**

1. Section 4.23, Power series, p. 192: # 23.8. This implies that uniform limits in general do not commute with taking derivatives.
3. First define the exponential function by the formula \(e^x := \sum_{i=0}^{\infty} \frac{1}{k!} x^k\). What is the radius of convergence for this series? Prove that \(e^0 = 1\) and \((e^x)'(x) = e^x\). Then do the exercise 26.4.

**Topics:** Differentiable \(\Rightarrow\) continuous, differentiation rules: \(\text{const}'\), \((cf)'\), \((f + g)'\), \((fg)'\) (product rule), \((f/g)'\) (quotient rule), two equivalent definitions of limit, i.e. for the equality \(\lim_{x \to a} f(x) = L\), ...

**For extra fun:**

- We defined the exponential function by the formula \(e^x := \sum_{i=0}^{\infty} \frac{1}{k!} x^k\). See an interesting discussion on the definition of the exponential function on page 341. Prove all the standard properties of the exponential function just from this definition.

Let me know what topics you are preparing for the projects. The dates for project presentations: Group 1 Monday, December 3; Group 2 Wednesday, December 5; Group 3 Friday, December 7.

**Homework 10. Due Friday, November 30.**

1. Section 5.29, The mean value theorem, p. 239: # 29.1*, 29.4*, 29.5, 29.8*, 29.9, 29.11, 29.13*, 29.14, 29.16*.

**Topics:** Chain rule, maximum at \(x_0\) and differentiable at \(x_0 \Rightarrow f'(x_0) = 0\), Rolle’s theorem, the statement of l’Hospital’s rule, the Riemann integral, its properties, ...

**Homework.**

1. ...