Math 403 Euclidean geometry, Fall 2017. Igor Mineyev.
Homework, topics, practice test, fun.

Below, “∗” means “turn in”, “no ∗” means “do not turn in, but know how to solve”. If a homework is in yellow color, it is still at a preliminary stage and might be modified later, but feel free to start working on it. I will also include an incomplete list of topics. Your practice test consists of:

- homework (listed below),
- topics (listed below),
- your class notes, and
- the textbook material covered in class.

Use them all as a guide to prepare for exams. If any questions arise, please ask me. Coming to office hours is also a very good idea. The problems marked “for extra fun” are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration.

Homework 1. Due Friday, September 8, 2017.

(1∗) Prove formally and rigorously that the eight properties of vectors in the plane stated in class indeed hold. Those are named (A1), (A2), (A3), (A4), (M1), (M2), (M3), (M4) in the book.

(2∗) (a) Write formally what it means for two sets $A$ and $B$ not to be equal.
    (b) Write formally what it means for two functions $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ not to be equal.
    (c) First define what it means for two vectors $X = \langle x_1, x_2 \rangle$ and $Y = \langle y_1, y_2 \rangle$ to be the same, i.e. $X = Y$. Then state what it formally means for these two vectors to be different, i.e. $X \neq Y$.

(3) Prove that $(\mathbb{R}^2, +)$ is a group.

(4) In analogy with problem (1), state eight properties for $\mathbb{R}^{100}$ instead of $\mathbb{R}^2$. Do these properties hold? Provide at least some justification for the answer.

(5) Chapter 1, page 5, exercises 1.2*, 1.3*, 1.4*, 1.5*. Make sure that you only use the eight properties or $\mathbb{R}^2$. For this problem, do not write vectors as pairs of numbers.

Topics: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, set, function, equality of sets, equality of functions, cartesian product, $\mathbb{R}^2$, the plane, a point in the plane, vector, group, abelian group, $\mathbb{R}^{100}$, injective, surjective, bijective, the standard (or abstract) line, the line passing through two (distinct!) points, equation of a line, equality of sets, equality of vectors, equality of functions, $\vec{AB}$, a basis in $\mathbb{R}^2$, general definition of proportionality of vectors, definition of a nonzero vector, proportionality of nonzero vectors, parallel lines, equivalence of statements, ...

For extra fun.

- How to get something out of nothing: Prove that the sets $\emptyset$ and $\{\emptyset\}$ are not equal.
- Prove that any two sets from the list

  $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\{\emptyset\}\}\}$, $\{\{\{\{\emptyset\}\}\}\}$, ...
are not equal.

- Prove that “the cartesian product of any two groups is a group”. It is also called the direct product of the two groups.
More precisely, let \((G, \cdot)\) and \((H, \cdot)\) be groups. (These two dots are meant to be different operations. To be precise, we should have denoted them, say, \('G\) and \('H\), but we won’t bother.) Define a “reasonable” binary operation \(\cdot\) on \(G \times H\) (still another operation which we could as well denote \(\cdot_{G \times H}\)) so that \((G \times H, \cdot)\) becomes a group. Prove that \((G \times H, \cdot)\) is indeed a group.

- Find your favorite group. Investigate what properties it has. Let me know what you found. Repeat.

Homework 2. Due Friday, September 15.
A general requirement for this class: whenever you divide by a number, say, \(a\), first explain why this number \(a\) is not zero. Whenever you use the notation \(l_{AB}\), first explain why \(A \neq B\).

1) Prove that for two nonzero vectors \(C\) and \(D\), the two definitions of proportionality given in class are equivalent.

2*) Let \(A\) and \(B\) be distinct points in the plane and define
\[
l_{AB} := \{A + t(B - A) \mid t \in \mathbb{R}\} = \{P \in \mathbb{R}^2 \mid \exists t \in \mathbb{R} \quad P - A = t(B - A)\}.
\]
Let \(C\) and \(D\) be two distinct points on the line \(l_{AB}\). Prove that \(l_{AB} = l_{CD}\). (\(l_{AB}\) and \(l_{CD}\) are sets. What does it mean for two sets to be equal?)

3*) Prove that for any two vectors \(B, C \in \mathbb{R}^2\), given explicitly as columns \(B = (b_1, b_2)\) and \(C = (c_1, c_2)\), the following statements are equivalent.
(a) \(B, C\) is a basis of \(\mathbb{R}^2\).
(b) Each equation of the form \(A = x_1B + x_2C\) has a unique solution \((x_1, x_2)\).
(c) \(b_1c_2 - b_2c_1 \neq 0\).
(This exercise does not assume any knowledge of linear algebra. If you use, say, Cramer’s rule or any other statement from linear algebra, prove it first.)

4*) Prove that for \(A, B, C \in \mathbb{R}^2\), the following statements are equivalent.
(a) \(A, B, C\) are not collinear.
(b) \(\{\overrightarrow{AB}, \overrightarrow{AC}\}\) is a basis of \(\mathbb{R}^2\).
(c) \(\{\overrightarrow{BA}, \overrightarrow{BC}\}\) is a basis of \(\mathbb{R}^2\).

5*) Let \(A\) and \(B\) be distinct points in the plane and \(C\) and \(D\) be distinct points in the plane. Prove that the following statements are equivalent.
(a) The lines \(l_{AB}\) and \(l_{CD}\) intersect at exactly one point.
(b) The vectors \(\overrightarrow{AB}\) and \(\overrightarrow{CD}\) are not proportional.

Topics: Triangle, parallelogram, informal discussion of isometries of objects, another definition of a line, another definition of a line, midpoint, median of a triangle, centroid(= center of mass) of a system of points, ...

For extra fun.
- Let \(\text{Isom}(\square)\) be the set of isometries of the square. Is \(\text{Isom}(\square)\) a group (in the sense of the formal definition of a group)? What is the operation? Is this group abelian?
Homework 3. Due Friday, September 22.

(1) Pages 7-9, exercises 1.6, 1.7*, 1.8*, 1.14.
(2) Page 9, exercise 1.9*. Compare the centers of $M_1M_2M_3M_4$ and $N_1N_2N_3N_4$.
(3) Page 11, exercise 1.10*.
(4*) For any triangle $\{A, B, C\}$, let $A', B', C'$ be the midpoints of the sides $[B, C]$, $[A, C]$, $[A, B]$, respectively.

(a) Show that the lines $l_{AA'}$, $l_{BB'}$, $l_{CC'}$ are well-defined.
(b) We proved in class that the medians $l_{AA'}$, $l_{BB'}$, $l_{CC'}$ have a common point. Show that this point is unique. [Hint: Consider, say, $C - G$ and $B - G$, where $G$ is the centroid of the triangle, and prove first that they are not proportional.]

Topics: Centroid of three points $\{A, B, C\}$ and the lines $l_{AA'}$, $l_{BB'}$, $l_{CC'}$, centroid of three points and the medians, relation between the centroid of a system of points and the centroids of its two parts, centroid of a system of mass-points, points in $l_{AB}$ as centroids of two mass-points, barycentric coordinates, the theorem of Ceva (precise statement), ...

For extra fun.

• Given a triangle $\{A, B, C\}$, characterize the median $l_{AA'}$ in terms of the moment of inertia of the system of points $\{A, B, C\}$ about various lines. How is the moment of inertia of $\{A, B, C\}$ about $l_{AA'}$ relates to the moments of inertia of $\{A, B, C\}$ about other lines passing through $A$?

Homework. To know before exam on Friday, September 29.

(0) If you have not done this yet, redo exercises 1.2, 1.3, 1.4, 1.5, making sure that you only use the eight properties of $\mathbb{R}^2$. For this problem, do not write vectors as pairs of numbers.
(1) Let $l_{AB}$ and $l_{CD}$ be lines in the plane. Prove that exactly one of the following conditions must hold.

(a) The two lines do not intersect, i.e. $l_{AB} \cap l_{CD}$ is the empty set.
(b) The intersection $l_{AB} \cap l_{CD}$ has exactly one point.
(c) $l_{AB} = l_{CD}$.
(2) Let $l = l_{AB}$ be a line and $C$ be a point in the plane not on the line $l$. Let $C'$ be any point in $l$. Prove that the intersection $l_{AB} \cap l_{CC'}$ consists of exactly one point. What is that point? (Use previous homework assignments if needed.)
(3) Given three points $A, B, C$ in the plane with $B \neq C$, write an equation of the line $l$ passing through $A$ and parallel to the line $l_{BC}$.
(4) Let $B$ and $C$ be distinct points in the plane. The interval $(B, C)$ is defined as the set

$$(B, C) := \{B + t(C - B) \mid t \in (0, 1)\}.$$

(a) Suppose $A' \in (B, C)$, $A' = bB + cC$, and $b + c = 1$. Prove that $b, c \in (0, 1)$.
(b) Suppose $b, c \in (0, 1)$, $b + c = 1$, and $A' = bB + cC$. Prove that $A' \in (B, C)$.

Topics: When and how the ratio of vectors is defined, the theorem of Ceva (complete and detailed proof), ...
For extra fun.

- Define what should be called barycentric coordinates in $\mathbb{R}^{100}$. State and prove a theorem about their existence and uniqueness. What should be the assumptions in that theorem?

Homework 4. Due Friday, October 6.

(1) Page 11, exercises 1.11*, 1.12*, 1.13*, 1.15*, 1.16*.

Topics: The theorem of Menelaus (complete and detailed proof), ...

For extra fun.

- We know the answer to the question: given a line $l$ in $\mathbb{R}^2$ and a point $A \in \mathbb{R}^2$, how many lines are there in $\mathbb{R}^2$ that pass through $A$ and are parallel to $l$? The answer to this question is the property called Euclid’s fifth postulate. Define your own “universe” (the formal term is a metric space) that has lines and define a notion of parallelism in such a way that Euclid’s fifth postulate does not hold. Such spaces go under various names: hyperbolic space, hyperbolic geometry, non-Euclidean space. An example is the hyperbolic plane $\mathbb{H}^2$; on the left is its artistic rendering by Escher (see www.mcescher.com/gallery/)

- Let $G$ be the set of all bijections $\mathbb{H}^2 \to \mathbb{H}^2$ that map angels to angels and devils to devils. (We also require the bijections to be continuous both ways.) Prove that $G$ is a group. Is $G$ abelian? What other interesting properties does $G$ have?

- Let $X$ be the Escher’s picture on the right. What kind of geometry does it represent? Let $G$ be the set of all bijections $X \to X$ continuous both ways that map each fish to a fish of the same color. Prove that $G$ is a group. Is $G$ abelian? What other interesting properties does $G$ have?

- Pick your favorite graph $\Gamma$; it consists of vertices and edges attached to vertices. Let $H$ be the set of all bijections $\Gamma \to \Gamma$ that map the vertices onto the vertices and edges onto the edges, and preserve the structure of the graph $\Gamma$. Prove that $H$ is a group with respect to the composition operation. Investigate, how many elements $H$ has. Is $H$ abelian? Does $H$ have any other interesting properties?

Homework 5. Due Friday, October 13.

(1*) Two lines $l_{AB}$ and $l_{CD}$ are called parallel if the (nonzero) vectors $B - A$ and $D - C$ are proportional. Show that this notion of being parallel is well defined, i.e. it depends only on the lines, and not on particular points on the lines. Specifically, prove that if
$A', B' \in l_{AB}$ are distinct points and $C', D' \in l_{CD}$ are distinct points, then the vectors $B - A$ and $D - C$ are proportional if and only if $B' - A'$ and $D' - C'$ are proportional.

(2*) Suppose that lines $l_{AB}$ and $l_{CD}$ are parallel and have a common point $P$. Prove that $l_{AB} = l_{CD}$.

(3) Page 20, exercises 1.17*, 1.18*, 1.20*.

(4) Let $\{A, B, C\}$ be a triangle (in general position, as always), $A' \in (B, C)$, $B' \in (A, C)$, $C' \in (A, B)$. Prove that
\[ \frac{|A' - B|}{|A' - C|} \frac{|B' - C|}{|B' - A|} \frac{|C' - A|}{|C' - B|} = 1. \]
Here $|X|$ denotes the length of a vector $X$. This is a weaker version of the theorem of Ceva, and it follows from the theorem of Ceva proved in class. In this exercise, do not use the theorem of Ceva proved in class. Rather, assume the formula for the area of a triangle, and use areas of triangles for the proof.

**Topics:** The theorem of Desargues (complete statement for now), the theorem of Pappus (complete statement for now), translations, central dilatations, ...

**For extra fun.**

- Prove Yuchen’s conjecture (suggested by him in class): Given a circle and a point $P$ outside of the circle. Let $A$ and $B$ be the points on the circle that belong to the two lines though $P$ tangent to the circle. Let $l$ be a line trough $P$ that intersects the circle at two points $C$ and $D$. Let $l'$ be another line trough $P$ that intersects the circle at two points $C'$ and $D'$ (in the same order). Let $X$ be the point of intersection of the lines $l_{C'D}$ and $l_{CD'}$. The conjecture says that $X$ belongs to the line $l_{AB}$. Prove or disprove the conjecture.

- The game Euclidea at \[\text{www.euclidea.xyz}\] It does keep you busy drawing tons of lines, circles, shapes in the plane. The game does not ask you to prove anything, so it is not a replacement for a course. But it might help you explore shapes, come up with your own conjectures, and to guess how to prove them.

Below are just some randomly chosen topics that you might consider for your project, they are intended as a guide only. Do your own research, suggest your own topic. Find what you like. Discuss your possible topic with me in advance.

(1) Conformal transformations of the plane, of the disc, of the half-space.
(2) What is the shape of the universe?
(3) Lorentzian geometry, space-time.
(4) Knot theory, a part of topology.
(5) Topology of manifolds, geometry of manifolds.
(6) See that Russian article with many triangles, lines and circles.
(7) Königsberg bridges, graph theory, topology.
(8) Flat spaces, negative curvature, positive curvature. Is the universe curved?
(9) Group actions on various geometric objects.
(10) Graphs, groups, Cayley graphs, how to visualize groups geometrically.
(11) Geometry in art, geometric patterns and their mathematical meaning.
(12) The fundamental group of a topological space, definition, many examples.
(13) Patterns, tessellations of the plane.
(14) Inner product on a vector space (aka dot product), Hilbert spaces.
(15) Riemannian structure, Riemannian manifolds, tangent vectors, the intrinsic metric (= path metric) on a manifold.
(16) Euclidean game project? www.euclidean.xyz Make sure that it consists not only of pictures, there must be a true mathematical component in it.
(17), (18), (19), .....................................................

Homework 6. Due Friday, October 20.

(1) Page 28, exercise 1.22*, 1.23*, 1.24*.
(2*) Let $V$ be any set and $S(V)$ be the set of all bijections from $V$ to itself. Prove that $(S(V), \circ)$ is a group. This group is called the symmetric group of the set $V$.
(3*) Prove that the set of all translations, $\{\tau_A : \mathbb{R}^2 \to \mathbb{R}^2 \mid A \in \mathbb{R}^2\}$, is a subgroup of $S(\mathbb{R}^2)$. (This implies, in particular, that the set of all translations is a group itself.)
(4) Page 34, exercise 2.1.

Topics: The symmetric group $S(V)$, subgroup, subgroup test, the (sub)group of translations of the plane, relations between translations and central dilatations, mapping lines to lines, a group of transformations of the plane (different from the book), ...

For extra fun. Prove the following subgroup test theorem:
If $(G, \cdot)$ is a group and $H$ is a nonempty (!) subset of $G$, then the following two statements are equivalent.

1. $H$ is a subgroup of $G$.
2. (a) $\forall x, y \in H \quad x \cdot y \in H$, and
   (b) $\forall x \in H \quad x^{-1} \in H$.

(Here $\cdot$ is the operation in $G$ and $x^{-1}$ denotes the inverse of $x$ in $G$.)

Homework. To know before exam, Friday, October 27.

(1) Prove that the set of all central dilatations centered at the origin
$$\{\delta_r : \mathbb{R}^2 \to \mathbb{R}^2 \mid r \in \mathbb{R} \setminus \{0\}\},$$
is a subgroup of $S(\mathbb{R}^2)$.
(2) Let $\alpha$ be a dilatation (i.e. translation or a central dilatation) and $l_{PQ}$ be any line.
   (a) Prove that $\alpha(l_{PQ}) = l_{\alpha(P), \alpha(Q)}$. (Equality of sets.)
   (b) Deduce that $\alpha(l_{PQ})$ is a line.
   (c) Prove that this line $\alpha(l_{PQ})$ is parallel to $l_{PQ}$.
(3) Page 37, exercises 2.2, 2.3.

Topics: Dilatation, the (sub)group of dilatations, dot product (= scalar product, inner product) and its properties, the length of a vector, circle $C(A, r)$, ...

For extra fun.

• Look up the definition of the semidirect product of two groups. Is the group of dilatations, $D(\mathbb{R}^2)$, a semidirect product in any way?
Look up the definition of a group action on a set. Does the group $D(\mathbb{R}^2)$ act on anything in any way?

**Homework 7. Due Friday, November 3.**

1. Page 39, exercises 2.4*, 2.5*, 2.6*.
2. Page 46, exercises 2.12, 2.13, 2.14*, 2.15, 2.16, 2.17.
3. List all the permutations of the set \{A, B, C, D\} that correspond to rigid motions of the square.
4. Define the permutations $\sigma$ and $\rho$ as on page 55. Explicitly describe each of the following permutations: $id, \rho, \rho^2, \rho^3, \sigma, \sigma \rho, \sigma \rho^2, \sigma \rho^3$. Here “product” means composition; say, $\sigma \rho^2$ means $\sigma \circ \rho \circ \rho$. How do these permutations compare to those in problem (3)?

**Topics:** Distance in the plane, perpendicular bisector (definition and equivalent restatement), perpendicular bisectors in a triangle, circumcenter, existence of circumcenter, ...

**For extra fun.**

- List all the permutations corresponding to all the rigid motions of an equilateral triangle.
- Define two permutations $\sigma$ and $\rho$, and make a list as in problem (4) above.
- Do the same for a regular pentagon, hexagon, etc.
- Describe all groups of order 1, of order 2, of order 3, of order 4, up to an isomorphism. (See exercises at the end of section 2.)