On quiver Hecke, quantum shuffle, and quantum cluster characters

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Motivation and Background. KLR-algs discovered by Khovanov-Lauda and Rouquier.

Classically, quantum affine Kac-Moody algebras categorify the upper half of a Lie group: $\mathbb{C}[N] \xrightarrow{\sim} \mathbb{C} \otimes K(\mathcal{C}')$, where $\mathcal{C}' \subseteq \mathcal{C}$ is a monoidal subcategory of the category of finite dimensional representations of quantum affine, and Lusztig's canonical basis of $\mathbb{C}[N]$ is mapped to the natural basis of $K(C')$ of simple objects.
Similarly, KLR-algebras categorify the lower half of the quantum group associated to the Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$:

Theorem ([KL11], Thm 8)

Let $U_q(\mathfrak{g})$ be a quantum group and let $R = \bigoplus_{n \geq 0} R(n)$ together with $(A, P, \Pi, P^\vee, \Pi^\vee)$ and $(Q_{ij})_{i,j\in I}$ be a quiver Hecke algebra (see below). Then $U^-_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g}) \xrightarrow{\sim} \bigoplus_{\beta \in Q^+} K(R(\beta) - \mathrm{Proj})$ is an $\mathbb{Z}[q^{\pm 1}]$-algebra isomorphism.

Some connections of KLR-algebras to other aspects of mathematics:

- Generalizing invariant theory of quivers to that of filtrations of vector spaces: [DW00], [DZ01], [SvdB01]
- Crawley-Boevey, Lusztig, Nakajima's quiver varieties and equivariant geometry: [GG06], [Gin09], [Nak04]
- Four dimensional quantum field theory and categorification: [CF94]
- Canonical and crystal bases of Ringel-Hall algebras: [Rin93]
- Geometric realization as the Ext-algebras of complexes of constructible sheaves on the moduli stack: [Rou08], [VV11]
- Generalized Grothendieck-Springer (symplectic) resolutions and quiver flag varieties: [CG10], [Im14a], [Nev11]
- Homotopy theory and invariant theory of quiver Hecke algebras (observation due to Ian Morrison at Fordham)

Results.

Definitions. Let $\mathbb{C}Q$ be a path algebra, and let (Q, I) be a quiver with relations, where Q is a quiver and $I \subseteq \mathbb{C}Q$ is a two-sided ideal generated by relations. A *quotient algebra* $\mathbb{C}Q/I$ is the path algebra of (Q, I). A path p is *reduced* if $0 \neq [p] \in \mathbb{C}Q/\langle q^2 : q \in \mathbb{C}Q, \mathrm{leng}(q) \geq 1\rangle$, where $\mathrm{leng}(q)$ is the number of arrows in q. A *pathway* from vertex i to vertex j is a reduced path from i to j, and *pathways* of Q is the set of all pathways from i to j as i, j vary over Q_0.

Flags of vector spaces: let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and let $Q = (Q_0, Q_1)$ be a quiver. Let $v = (v^1, \ldots, v^n) \in (\mathbb{Z}^{Q_0}_{\geq 0})^n$ and let $F^\bullet : 0 = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^n$ be a sequence of representations of Q such that $\mathrm{rk}(V^r/V^{r-1}) = v^r$. Define $\alpha := \sum_r v^r$ and $\alpha_i := \sum_{r=1}^{n} v_i^r$. Let $Rep_\alpha := \prod_{h:i\to j} \mathrm{Hom}_{\mathbb{C}}(V^{\alpha_i}, V^{\alpha_j})$, where $h \in Q_1$ and $\dim V^{\alpha_i} = \alpha_i$. Let $F^\bullet Rep_\alpha \subseteq Rep_\alpha$ preserve $F^\bullet$, and define $\mathbb{G}_\alpha := \prod_{i\in Q_0} GL(V^{\alpha_i})$ and $\mathbb{P}_\alpha := \prod_{i \in Q_0} P_{\alpha_i}$, which fixes $F^\bullet$, where P_{α_i} is a subgroup of $GL(V^{\alpha_i})$ for all $i \in Q_0$.

Theorem ([Im14b], Thm 1.1)

Let $F^\bullet$ be $v = (1, 1, \ldots, 1) \in (\mathbb{Z}^{Q_0}_{\geq 0})^n$. Then Q is a quiver with at most two distinct pathways between any two vertices if and only if $\mathbb{C}[F^\bullet Rep_\alpha]^{\mathbb{U}_\alpha} \cong \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$.

Corollary (-): if $F^\bullet$ corresponds to $v^r = (k_r, \ldots, k_r) \in \mathbb{Z}^{Q_0}_{\geq 0}$ for each r, then Q has at most two distinct pathways between any two vertices if and only if $\mathbb{C}[F^\bullet Rep_\alpha]^{\mathbb{U}_\alpha} \cong \mathbb{C}[\mathfrak{l}^{\oplus Q_1}]$.

Corollary (-): above Corollary holds for ADE-Dynkin finite and untwisted affine type quivers.

Quiver Hecke algebras, following [KKK14], most general definition.

Other definitions are in [KL11] and [Rou12]. Let $\mathbf{k}$ be a field, and let I be a finite index set. Let A be a Cartan matrix, P is the weight lattice, Π is the simple roots, $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$ is the dual weight lattice, $\Pi^\vee = \{h_i : i \in I\} \subseteq P^\vee$ is the simple coroots such that $\langle h_i, \alpha_j\rangle = a_{ij}$ for all $i, j \in I$, Π is linearly independent, and for all $i \in I$, there exists $\Lambda_i \in P$ such that $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ for all $j \in I$. For $i \neq j \in I$, let $S_{i,j} := \{(p, q) \in \mathbb{Z}^2_{\geq 0} : (\alpha_i|\alpha_i)p + (\alpha_j|\alpha_j)q = -2(\alpha_i|\alpha_j)\}$. Define polynomials in the matrix $(Q_{ij})_{i,j\in I}$ in $\mathbf{k}[u, v]$ as $Q_{ij}(u, v) = \sum_{(p,q)\in S_{i,j}} t_{i,j;p,q} u^p v^q$ if $i \neq j$, or $Q_{ij}(u, v) = 0$ if $i = j$. Remark: $Q_{ij}(u, v) = Q_{ji}(v, u)$ and $t_{i,j;-a_{ij},0} \in \mathbf{k}^*$ is a nonzero constant. Denote $S_n = \langle s_1, \ldots, s_{n-1}\rangle$ as the symmetric group on n letters, where $s_i = (i, i + 1)$ is a simple transposition, and S_n acts on I^n by place permutations.
KLR-algebras $R(n)$ of degree n associated with a Cartan datum and $(Q_{ij})_{i,j\in I}$ is the associative algebra over $\mathbf{k}$ generated by the elements $\{e(\nu)\}_{\nu\in I^n}$, $\{x_k\}_{1\leq k\leq n}$, $\{\tau_m\}_{1\leq m\leq n-1}$ that satisfy the following relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \ \sum_{\nu\in I^n} e(\nu) = 1, \ x_k x_m = x_m x_k, \ x_k e(\nu) = e(\nu)x_k, \ \tau_m e(\nu) = e(s_m(\nu))\tau_m, \ \tau_k\tau_m = \tau_m\tau_k \text{ if } |k - m| > 1,$$

$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu), \ (\tau_k x_m - x_{s_k(m)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } m = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } m = k + 1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \dfrac{Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) & \text{if } \nu_k = \nu_{k+2} \neq \nu_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

$R(n)$ is a $\mathbb{Z}$-graded algebra with grading: $\deg(e(\nu)) = 0$, $\deg(x_k e(\nu)) = (\alpha_{\nu_k}|\alpha_{\nu_k})$, $\deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}|\alpha_{\nu_{l+1}})$. For $n \geq 0$ and $\beta = \sum_{i\in I} m_i\alpha_i \in Q^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ such that $\mathrm{height}(\beta) = \sum_{i\in I} m_i = n$, $I^\beta := \{\nu = (\nu_1, \ldots, \nu_n) \in I^n : \alpha_{\nu_1} + \ldots + \alpha_{\nu_n} = \beta\}$ and $R(\beta) = \bigoplus_{\nu\in I^\beta} R(n)e(\nu)$ is the *KLR-algebra at β*.

Quantum cluster algebras, following [KQ14] and [Rup15].

Let Q be an acyclic quiver, $D = \mathrm{diag}(s_i : i \in I)$ be the symmetrizing matrix, and $n_{ij} = n_{ji}$ is the number of arrows connecting vertices $i, j \in I$ in Q. Define adjacency matrix $B = B_Q = (b_{ij})$ as follows: $b_{ij} = n_{ij}s_j/\gcd(s_i, s_j)$ if $i \to j$ in Q, $b_{ij} = -n_{ij}s_j/\gcd(s_i, s_j)$ if $j \to i$ in Q, and $b_{ij} = 0$ if $i = j$. Consider an index set $J \supseteq I$ and a $J \times I$ matrix $\widetilde{B} = (b_{ij})$ with principal $I \times I$ submatrix B. Let $\Lambda = (\lambda_{ij})$ be a $J \times J$ matrix compatible with $\widetilde{B}$: $\sum_{i\in J} \lambda_{ij}b_{ik} = \delta_{jk}s_k$ for all $j \in J$ and $k \in I$. The *quantum torus* $\mathcal{T}_{\Lambda,q}$ is generated by $\mathbf{X} = (X_i : i \in J)$ with relations $X_iX_j = q^{\lambda_{ij}}X_jX_i$ for $i, j \in J$. Let $\Sigma_0 = (\mathbf{X}, \widetilde{B})$ be a *quantum seed*, $\mathbf{X}$ the *cluster*, and $\widetilde{B}$ the *exchange matrix*. Imposing a total ordering $<$ on J, for $\mathbf{a} \in \mathbb{Z}^J$, define $X^{\mathbf{a}} := q^{1/2\sum_{i<j} a_ia_j\lambda_{ji}} \overrightarrow{\prod_{i\in J}} X_i^{a_i} \in \mathcal{T}_{\Lambda,q}$, where $\overrightarrow{\prod}$ denotes the product in increasing order. The *mutation $\mu_k\Sigma = (\mu_k\mathbf{X}, \mu_k\widetilde{B})$ in direction k* is defined as: $\mu_k\mathbf{X} = (\mathbf{X} \setminus \{X_k\}) \cup \{X'_k\}$, where $X'_k = X^{\mathbf{b}^k_+ - \alpha_k} + X^{\mathbf{b}^k_- - \alpha_k}$, with $\mathbf{b}^k_\pm = \sum_{i=1}^{|I|} \max\{0, \pm b_{ik}\}e_i$ such that $\mathbf{b}^k_+ - \mathbf{b}^k_- = \mathbf{b}^k$ is the k-th column of $\widetilde{B}$, and $\mu_k\widetilde{B} = (b'_{ij})$ with entries:

$$b'_{ij} = \begin{cases} b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0, \end{cases} \qquad \begin{matrix} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} & \text{otherwise.} \end{matrix}$$

Let $\mathbb{T}$ be a rooted, labelled I-regular tree with root vertex t_0 with edges emanating from each vertex labelled by distinct elements of I. Assign quantum seeds $\Sigma_t = (\mathbf{X}_t, \widetilde{B}_t)$ to the vertices $t \in \mathbb{T}$ such that $\Sigma_{t_0} = \Sigma_0$ and for $t \overset{k}{-} t'$ in $\mathbb{T}$, the seeds Σ_t and $\Sigma_{t'}$ are related by the mutation in direction k. *Quantum cluster algebra* is the $\mathbb{Z}[q^{\pm 1/2}][\![X_j^{\pm 1} : j \in J \setminus I]\!]$-subalgebra of the quantum torus generated by all cluster variables from all seeds mutation equivalent to Σ_0: $\mathbb{Z}[q^{\pm 1/2}](\mathbf{X}, \widetilde{B}) = \mathbb{Z}[q^{\pm 1/2}][\![X_j^{\pm 1} : j \in J \setminus I]\!][X_{i;t} : t \in \mathbb{T}, i \in I] \subseteq \mathcal{T}_{\Lambda,q}$.

Quantum shuffle algebras, following [Lec04].

Other constructions are given by Kleshchev-Ram in [KR08] and Dylan Rupel in [Rup15]. Recall q-derivations e'_i, $1 \leq i \leq r$, of $U_q(\mathfrak{n})$ by [Kas91]: $e'_i \in \mathrm{End}(U_q(\mathfrak{n}))$ such that $e'_i(e_j) - \delta_{ij}$, $e'_i(uv) = e'_i(u)v + q^{-(\alpha_i,|u|)}ue'_i(v)$ for all homogeneous $u, v \in U_q(\mathfrak{n})$, where $|u| = \deg_{Q^+}(u)$. Note $e'_i(u) = 0$ for all $1 \leq i \leq r$ iff $|u| = 0$, and that e'_i satisfy the q-Serre relations. Furthermore, e'_i is the endomorphism adjoint to left multiplication by e_i: $\exists\,!$ nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $U_q(\mathfrak{n})$ such that $(1, 1) = 1$ and $(e'_i(u), v) = (u, e_iv)$, $u, v \in U_q(\mathfrak{n})$, $1 \leq i \leq r$.
Let $\mathcal{M}$ be a free monoid generated by $I = \{w_1, \ldots, w_r\}$, and let $\mathcal{F}$ be a free associative algebra over $\mathbb{Q}(q)$ generated by I. Write $w[i_1, \ldots, i_k] := w_{i_1} \cdots w_{i_k}$, $w[\ ]$ as the empty word, and $l(w) :=$ length of w. $\mathcal{F}$ is Q^+-graded by $\deg(w_i) = \alpha_i$. Associate $\partial_w := e'_{i_1} \cdots e'_{i_k} \in \mathrm{End}(U_q(\mathfrak{n}))$ to each $w = w[i_1, \ldots, i_k]$ and $\partial_w = \mathrm{Id}_{U_q(\mathfrak{n})}$ to $w = w[\ ]$. If u is a homogeneous element of $U_q(\mathfrak{n})$ and $|w| = |u|$, then $\deg(\partial_w(u)) = 0$. Let $\Phi : U_q(\mathfrak{n}) \to \mathcal{F}$, $\Phi(u) = \sum_{w\in\mathcal{M},|w|=|u|} \partial_w(u)w$ for homogeneous $u \in U_q(\mathfrak{n})$.
Next, we inductively define a bilinear map $*$ from $\mathcal{F}$ to $\mathcal{F}$: for $a, b \in I$ and $w, x \in \mathcal{M}$,

$$wa * xb = (w * xb)a + q^{-(|wa|,|b|)}(wa * x)b, \qquad w[\ ] * x = x * w[\ ] = x.$$

Iterating, we get

$$w[i_1, \ldots, i_m] * w[i_{m+1}, \ldots, i_{m+n}] = \sum_\sigma q^{-e(\sigma)} w[i_{\sigma(1)}, \ldots, i_{\sigma(m+n)}],$$

where $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m + 1) < \cdots < \sigma(m + n)$, $\sigma \in \mathcal{S}_{m+n}$, and

$$e(\sigma) = \sum_{k\leq m<l;\sigma(k)<\sigma(l)} (\alpha_{i_{\sigma(k)}}, \alpha_{i_{\sigma(l)}}).$$

Connecting the three algebras. Current development and open problems.

Let $K(R(\beta)-\mathrm{gmod})$ be the Grothendieck group of finite dimensional $\mathbf{k}$-graded $R(\beta)$-modules. Define $K(R-\mathrm{gmod}) \xrightarrow{\mathrm{ch}_q} \mathcal{F}$, $[M] \mapsto \mathrm{ch}_q(M) = \sum_{w\in\mathcal{M}} \dim_q e(w)M \cdot w$ as the *KLR-algebra character.* Given a finite dimensional graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$, we write $\dim_q V = \sum_{n\in\mathbb{Z}}(\dim V_n)q^n$ for its *graded character.*
Let $\mathcal{H}^*(Q, \mathbf{d})$ be the graded dual Hall-Ringel bialgebra. *Quantum shuffle character* $\Omega : \mathcal{H}^*(Q, \mathbf{d}) \to \mathcal{F}$ is defined as follows: for a word $w = w[i_1, \ldots, i_k]$, write $\mathcal{F}_w(V) := \{0 = V_k \subseteq V_{k-1} \subseteq \cdots \subseteq V_1 \subseteq V_0 = V : V_{r-1}/V_r \cong S_{i_r} \text{ for } 1 \leq r \leq k\}$, where S_{i_r} is a simple representation. Then a basis vector $[V]^* \in \mathcal{H}^*(Q, \mathbf{d})$ is mapped to a generating function for counting flags in V: $\Omega([V]^*) = \sum_{w\in\mathcal{M}} q^{-\sum_{k<l}\langle\alpha_{i_l},\alpha_{i_k}\rangle}|\mathcal{F}_w(V)| \cdot w$.
Result known to date: for symmetric Cartan matrices, [VV11] denotes $\widetilde{F}_{\widetilde{V}}$ as the variety of pairs and proves that the character (its Poincaré polynomial) of $H_*^{G_{\widetilde{V}}}(\widetilde{F}(V))$ in $\mathcal{F}$ is exactly the counting polynomial $\Omega([V]^*)$.
Open: relate KLR-algebra characters and quantum shuffle characters for symmetrizable Cartan matrices.
Next, it has been known that quantum cluster monomials are contained in the dual canonical basis of quantum coordinate ring.

Let $\mathbf{i} \in I^m$ be a reduced word which correspond to Weyl group element $w = s_{i_1} \cdots s_{i_m} \in W$. Let $\Psi_{\mathbf{i}} : \mathbb{Z}[q^{\pm 1/2}] \to P_{\mathbf{i}}$, a quantum polynomial ring, where $\Psi_{\mathbf{i}}(x_j) = \sum_{k:i_k=j} t_k$ for $j \in I$. Then $\mathbb{Z}[q^{\pm 1/2}][\![N \cap B_-wB_-]\!] \cong \mathbb{Z}[q^{\pm 1/2}][\![N]\!]/\ker\Psi_{\mathbf{i}}$. Open: find a q-cluster structure on this algebra such that the cluster monomials corresp. to irreducible KLR-rep characters.

For further investigation.

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