From a quantum field theory to categorical representation theory

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Figure: Quantum entanglement, http://goo.gl/mqquTc

Motivation.

Why study topological quantum field theory (TQFT)? It computes topological invariants. It is also related to knot theory, 4-manifolds in algebraic topology, moduli spaces in algebraic geometry.

Category theory is a way of recognizing constructions that appears when you are talking about different kinds of mathematical objects that have something in common.

Example: cartesian product of abelian groups, cartesian product of manifolds, etc. Although the objects are different, the general notion of cartesian product is the same.

TQFT produces a tower of algebraic structures, each dimension related to the previous one by the process of *categorification*.

Remark: some higher dimensional theories exist but are not well-understood.

Background.

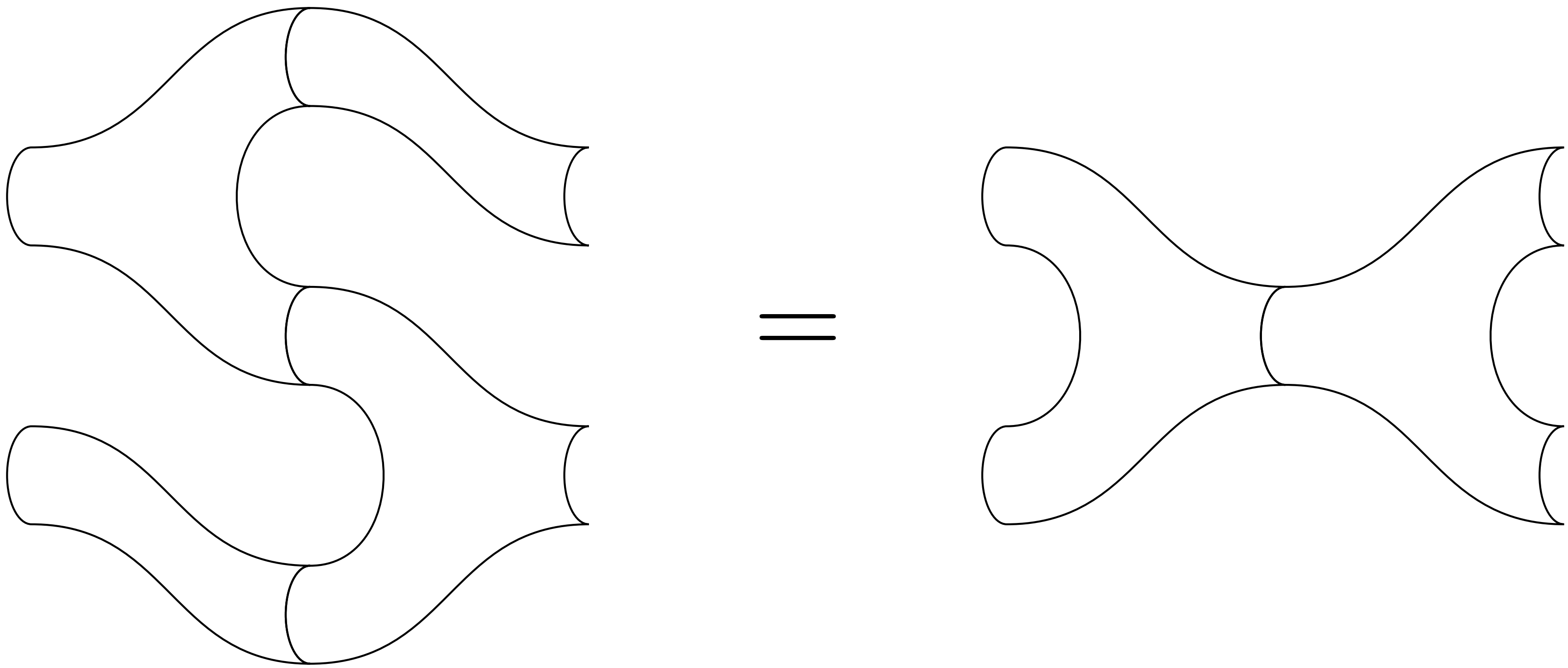
Definition: a *manifold* is a smooth, compact, (oriented) topological space.
Note: manifolds may have boundaries.
Definition: a *closed manifold* is a manifold without a boundary.
Examples:

- a sphere
- a donut
- a notebook paper (a 2-manifold with 1-dimensional boundary)
- the boundary of an n-manifold is an $(n-1)$-manifold.

Organize closed manifolds with a fixed dim as a category $\mathrm{Cob}(d)$:

- objects: closed $(d-1)$-manifolds,
- morphisms: $\mathrm{Hom}(M, N) \simeq \{B : \partial B \simeq \overline{M} \coprod N\}/\,\mathrm{diffeom}$, where $\overline{M}$ is M in opposite orientation.

Two bordisms are the same in $\mathrm{Cob}(d)$ if they are diffeomorphic relative to their boundary, and composition is given by gluing the morphisms.



What is a category?

Category theory is an abstract branch of mathematics with many applications, including programming.

Introduced by S. Eilenberg and S. MacLane in 1945 [EM45].

Definition

A *category* $\mathcal{C}$ consists of

- a class $\mathrm{ob}(\mathcal{C})$ of objects: $x, y, z, \ldots \in \mathrm{ob}(\mathcal{C})$,
- a class hom$(\mathcal{C})$ of morphisms or arrows or maps between the objects: $(x \to y) \in \mathrm{hom}(\mathcal{C})$,
- composition of morphisms: given x, y, $z \in \mathcal{C}$ and $x \to y$ and $y \to z$, then $(x \to z) \in \mathrm{hom}(\mathcal{C})$,
- an identity morphism for every object: $(x \xrightarrow{\mathrm{id}} x) \in \mathrm{hom}(\mathcal{C})$ for each $x \in \mathrm{ob}(\mathcal{C})$,
- associativity of morphisms: if $x \xrightarrow{f} y$, $y \xrightarrow{g} z$, and $z \xrightarrow{h} w$ are morphisms in the category, then $h \circ (g \circ f) = (h \circ g) \circ f$.

What is TQFT?

Definition ([Ati88b], [Wit88])

TQFT of dimension d is the symmetric, monoidal functor

$$Z : \mathrm{Cob}(d) \longrightarrow \mathrm{Vect}_{\mathbb{C}},$$

which preserves tensor products $\otimes$.

The $\otimes$ in $\mathrm{Cob}(d)$ is given by disjoint union of manifolds while $\otimes$ in $\mathrm{Vect}_{\mathbb{C}}$ is given by the tensor product of vector spaces:

$$Z(M \coprod N) = Z(M) \otimes Z(N), \quad Z(\varnothing) \simeq \mathbb{C},$$

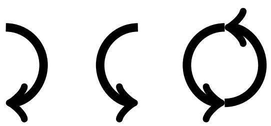
where $\mathbb{C}$ is a unit with respect to the tensor product on $\mathbb{C}$-vector spaces.

Example: $\mathrm{Cob}(1)$. Let $d = 1$.

Objects are 0-dimensional manifolds with orientation: $\overset{+}{\bullet}$, $\overset{-}{\bullet}$.

$Z\left(\overset{+}{\bullet}\right) = X$ finite dimensional, where $\overset{+}{\bullet}$ has positive orientation.

$Z\left(\overset{-}{\bullet}\right) = Y$ finite dimensional, where $\overset{-}{\bullet}$ has negative orientation.



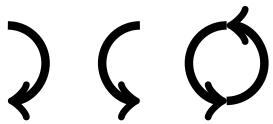
Caption: first image A, second image B, third image C.

$Z(A)$: $X \otimes Y \longrightarrow \mathbb{C}$

$Z(B)$: $\mathbb{C} \longrightarrow Y \otimes X$

The maps $Z(A)$ and $Z(B)$ exhibit that X and Y must be duals of each other (cf. [Lur09], Prop 1.1.8): $Y \simeq X^{\vee}$. In particular, both are finite dimensional vector spaces.

Example: $\mathrm{Cob}(1)$ continued.


Caption: first image A, second image B, third image C.

What does the functor Z do to objects?

$$Z\left(\overset{+}{\bullet} \coprod \overset{-}{\bullet} \coprod \overset{-}{\bullet}\right) = X \otimes X^{\vee} \otimes X^{\vee}.$$

What does the functor Z do to morphisms?

$$Z\left(+ \longrightarrow +\right) = \mathrm{id}_X, \qquad Z\left(- \longrightarrow -\right) = \mathrm{id}_{X^{\vee}}.$$

$$Z(A) = (X \otimes X^{\vee} \xrightarrow{\mathrm{ev}} \mathbb{C}),\ Z(B) = (\mathbb{C} \xrightarrow{\mathrm{coev}} X \otimes X^{\vee})$$

$Z(C)$:

$$\mathbb{C} = Z(\varnothing) \overset{Z(B)=\mathrm{id}}{\hookrightarrow} Z(\overset{-}{\bullet}\ \overset{+}{\bullet}) = X \otimes X^{\vee} = \mathrm{End}(X) \xrightarrow{Z(\overline{A})=\mathrm{tr}} Z(\varnothing) = \mathbb{C}$$

imply $Z(C) = \dim X$, where $\overline{A}$ is A with an opposite orientation.

Example: $\mathrm{Cob}(1)$ continued.

Conclusion: 1-dimensional TQFT is determined by what it does to a single point, i.e., it is determined by a single vector space.

One can evaluate field theory on other manifolds to get invariants. For example, evaluate field theory on a circle to get the only invariant on a complex vector space, which is its dimension:

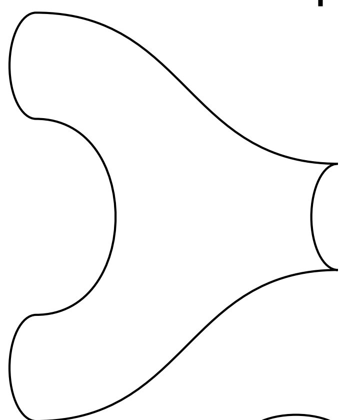
a complex vector space is determined up to isomorphism by its dimension.

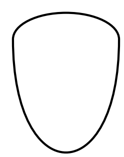
By turning the vector space into a field theory, then the invariant associated to the only interesting 1-manifold is the dimension of X.

Diagrammatic maps for $\mathrm{Cob}(2)$.

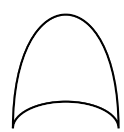
We read diagrammic objects from left to right or from bottom to top.

Pair of pants:





Cup



Cap

Example: $\mathrm{Cob}(2)$, [Abr96], [Dij89], [Koc04], [Saw95].

Objects are closed manifolds of dimension 1, i.e., circles S^1.

$Z(S^1) = A$, a complex vector space, $Z(S^1 \coprod S^1) = A \otimes A$.

Z(pair of pants): $A \otimes A \xrightarrow{m} A$, a multiplication map, where m is associative and commutative.

Z(cup): is the bordism $\varnothing \longrightarrow S^1$, $\mathbb{C} \simeq Z(\varnothing) \longrightarrow Z(S^1) = A$.

$Z(\text{cup}) = 1 \in A$ is a unit with respect to the multiplication.

Z(cap): $A \xrightarrow{\mathrm{tr}} \mathbb{C}$, is the comultiplication map.

Since the trace pairing is nondegenerate:

$$A \otimes A \xrightarrow{m} A \xrightarrow{\mathrm{tr}} \mathbb{C}, \quad Z(S^1 \coprod S^1) \xrightarrow{\text{comp}} Z(\varnothing),$$

we have $A \simeq A^\vee$. Again, we see that A is finite dimensional.

Example: $\mathrm{Cob}(2)$ continued.

Conclusion: start with 2-dimensional TQFT, extract algebraic structure to get a vector space that is commutative and associative, with nondegenerate trace. This algebraic structure A, which is commutative with nondegenerate trace, is called a Frobenius algebra.

Converse is also true: given any Frobenius algebra A, we can find a unique 2-TQFT up to isomorphism!

In $d = 2$ case, $Z(M)$ is a number you could extract from a commutative Frobenius algebra via cutting M into pieces, each of which is a disk or a pair of pants, and the behavior of our field theory on disks and a pair of pants is precisely what is encoded in this Frobenius algebra.

In order to compute the number $Z(M)$, write down the vector space associated to these cuts, and all the linear algebra maps in terms of the Frobenius algebra structure and compose them to see what number you get in the end.

A discussion of $Cob(d)$ for higher d.

For 2-manifold M,

- if $\text{genus}(M) = 0$, get $Z(M) = \text{tr}(1)$,
- if $\text{genus}(M) = 1$, get $Z(M) = \dim(A)$.

Special cases in $\text{Cob}(2)$ are related to string theory, lattice gauge theory, constructions using triangulations.

We saw what bordism does on manifolds of codimension 1: if you evaluate Z on an $n-1$ manifold, you get a vector space, which have some structure by evaluating Z on bordisms.

Another perspective: for d arbitrary and a closed d-manifold M, $Z(M)$: $\mathbb{C} \simeq Z(\varnothing) \xrightarrow{\lambda} Z(\varnothing) \simeq \mathbb{C}$, where λ is multiplication by a scalar. Thus $Z(M) = \lambda$, which is a diffeomorphism invariant of manifolds.

TQFT gives you a set of tools for computing these numerial invariants that are associated to manifolds of top dimension.

In higher dimensions, cut these manifolds by cutting them into small pieces via triangulation, with some care needed when gluing submanifolds along corners. In $\mathrm{Cob}(d)$, gluing should be given by composition along closed submanifolds of dimension $d-1$.

Defn: an *extended TQFT of dim d* is a rule associating data to:

- each closed d-manifold $\rightsquigarrow$ a complex number
- each closed $(d-1)$-manifold $\rightsquigarrow$ complex vector space
- each bordism of $(d-1)$-manifolds $\rightsquigarrow$ a linear map of vector spaces
- each closed $(d-2)$-manifolds $\rightsquigarrow$ $\mathbb{C}$-linear category, e.g., $\mathrm{Vect}_{\mathbb{C}}$
- each bordism of $(d-2)$-manifolds $\rightsquigarrow$ $\mathbb{C}$-linear functors
- each closed $(d-3)$-manifolds $\rightsquigarrow$ 2-categories
- each bordism $(d-3)$-manifolds $\rightsquigarrow$ linear transformations

Associate invariants to manifolds of dimension $d, d-1, d-2, \ldots$
Note: more precise definition in Appendix of the slides.

Example: $\mathrm{Cob}(3)$, [CF94], [CFS94], [Kup91].

Remark: for $d \gg 0$, manifolds have corners and look very complicated but locally, a manifold looks very simple like a Euclidean space.

The structure of finite dimensional complex Hopf algebras, e.g., quantum groups, could be expressed in 3-dimensional TQFT. Furthermore, 3-dimensional (polyhedral) triangulations lead to the identities of Hopf algebras.

Key idea: label the edges of the faces with algebra basis elements, combine around faces with the multiplication, around edges with the comultiplication, and sums over labelings. To define an invariant associated to this decomposition from a Hopf algebra, attach an integer to each edge, a half-integer and an orientation to each face, and an origin to each face ([CF94]).

Higher categories.

In higher categories, extract some formal features that makes sense in general categories.

In a strict 2-category,

- objects: categories
- morphisms: functors
- 2-morphisms: natural transformations

See Appendix for a brief discussion on ∞-categories.

Example: $\mathrm{Cob}(4)$. Khovanov homology and Jones polynomial, [Kho00], [Kho05].

Khovanov homology appears in a 4-dimensional TQFT, which is a categorical link invariant where

$$L \mapsto \mathrm{Kh}(L) \in \mathcal{C},$$

and if L is isotopic to L', then $\mathrm{Kh}(L) \cong \mathrm{Kh}(L')$, and we also have an isomorphism of cobordisms: $Z(L) \simeq Z(L')$.

Now, there is a map from $\mathrm{Kh}(L)$ to its representative $[\mathrm{Kh}(L)]$ in the Grothendieck group

$$\mathrm{Kh}(L) \mapsto [\mathrm{Kh}(L)] \in K(\mathcal{C}),$$

and if we identify $K(\mathcal{C}) = \mathbb{Z}[q^{\pm 1}]$, then $[\mathrm{Kh}(L)] = J(L)$, where $J(L)$ is the Jones polynomial of L.

Thus, Khovanov homology categorifies the Jones polynomial.

Categorical representation theory.

The idea of *categorification*: construct a tensor category or a 2-category whose Grothendieck group is isomorphic to a given algebraic structure.

Examples [Kan13]: *KLR*-algebras categorify the negative half of the corresponding quantum group, i.e.,

Theorem ([KL11])

Let $U_q(\mathfrak{g})$ be the quantum group associated with a Cartan datum and let $R = \bigoplus_{n \geq 0} R(n)$ be the KLR-algebra. Then

$$U^-_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g}) \xrightarrow{\sim} \bigoplus_{\beta \in Q^+} K(R(\beta) - \mathrm{Proj})$$

is an $\mathbb{Z}[q^{\pm 1}]$-algebra isomorphism.

We also have a cyclotomic categorification for cyclotomic quiver Hecke algebras $R^{\Lambda}(n)$ where Λ is a dominant integral weight:

Theorem ([KK12], Thm 6.2)

For $\Lambda \in P^{+}$, there is an isomorphism between

irreducible highest weight $U_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g})$-module $L(\Lambda)$ with dominant integral highest weight Λ and the Grothendieck group of finitely generated graded projective cyclotomic quiver Hecke algebra modules $\bigoplus_{\nu} K(R^{\Lambda}(\nu) - \mathrm{Proj})$.

Connections between TQFT and CRT.

Initially, there were no mathematical constructions of 4-dimensional TQFT (see [Ati88a] and [Wit88]), but some constructions emerged in Crane-Frankel's [CF94] and Crane-Yetter's [CY93].

Using Hopf categories, Crane-Frankel predicts a 4-dimensional TQFT, and later constructs a nontrivial example of a Hopf category from the upper half $\dot{U}_q^+(\mathfrak{sl}_2)$ of the modified quantum group, where objects of categorification correspond to Lusztig's canonical bases.

TQFT have been of interest to mathematicians as they are tools to construct topological invariants of manifolds.

Application of the categorification of quiver Hecke algebras to 4-dimensional TQFT? See [Bru13].

Reference on the origin and historical developments between TQFT and CRT: [BL11].

Appendix: supplementary material.

The following pages consist of more rigorous constructions and definitions.

Extended TQFT and strict n-categories.

Defn: an *extended TQFT* is a $\otimes$-functor between d-categories.

- d-category made up of manifolds $\rightsquigarrow$ $\mathbb{C}$-numbers
- bordisms between manifolds $\rightsquigarrow$ $\mathbb{C}$-vector spaces
- bordisms between bordisms $\rightsquigarrow$ $\mathbb{C}$-linear categories
- $\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

In all dimensions, an extended TQFT Z is determined by $Z(\bullet)$.
There is a more elaborative definition of non-strict n-category whose associativity is not required to hold, but is required to hold up to *coherent isomorphism*, i.e., paths are allowed to have variable lengths.

Defn: a *strict n-category* $\mathcal{C}$ consists of

- a collection of objects $x, y, z, \ldots$
- have $(n-1)$-Category of $\mathrm{Hom}_{\mathcal{C}}(x, y)$ for every pair of objects x and y
- composition functors: $\mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{C}}(y, z) \rightarrow \mathrm{Hom}_{\mathcal{C}}(x, z)$
- the composition law is associative with units, which holds up to coherent isomorphism.

A discussion on n-groupoids.

Let X be a topological space, $x \in X$ base point.
$\pi_1(X, x)$ is a group whose elements are paths in X that begin and end at x; it is a homotopy class of such paths. Composition of paths are given by concatenation of paths.

It we don't choose a base point x, can define $\pi_{\leq 1}X$ which is the fundamental groupoid of X. Cat whose objects are points of X, morphisms are homotopy classes of paths, a composition is a concatenation of paths.

$\pi_{\leq 1}X$ contains information about $\pi_0 X$ and $\pi_1 X$, but it doesn't know about the higher homotopy groups:

- $\pi_0 X$ is the set of path components of X, which is the set of isomorphism classes of objects in this groupoid.
- $\pi_1 X$ is the fundamental group of each connected component, which is the automorphism group of the corresponding objects.

Defn: An *n-groupoid* is n-category where all k-morphisms are invertible, for $1 \leq k \leq n$.

A discussion on n-groupoids and ∞-groupoids.

$\pi_{\leq n} X$ is a fundamental n-groupoid of X, where

- objects: points of X
- 1-morphisms: paths
- 2-morphisms: homotopies between paths
- 3-morphisms: homotopies between homotopies
- $\vdots$
- n-morphism: n-fold homotopies mod homotopy

Defn: $\pi_{\leq \infty} X$ is a fundamental ∞-groupoid where objects are points of X, 1-morphisms are paths, 2-morphisms are homotopies of paths, etc. At no level do we mod out by homotopies.

Note: an ∞-groupoid is a topological space.

Def: An (∞, n)-*category* is a higher category where all k-morphisms are invertible for $k > n$.

Example: $\mathcal{C}$ is an $(\infty, 0)$-category if and only if it is an ∞-groupoid if and only if it is a topological space.

Thank you.

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