Quiver Hecke algebras and filtered quiver representations

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Flags of representations ([CG10], [Nev11]) over $\mathbb{C}$

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and let $Q = (Q_0, Q_1)$ be a quiver. Let $\nu = (\nu^1, \ldots, \nu^n) \in (\mathbb{Z}_{\geq 0}^{Q_0})^n$ and let $F^\bullet : 0 = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^n$ be a sequence of representations of Q such that

$\mathrm{rk}(V^r/V^{r-1}) = \nu^r$. We let $\alpha := \sum_r \nu^r$ and $\alpha_i := \sum_{r=1}^n \nu_i^r$.

Let $Rep_\alpha := \prod_{h:i\to j} \mathrm{Hom}_{\mathbb{C}}(V^{\alpha_i}, V^{\alpha_j})$, where $h \in Q_1$, $\dim V^{\alpha_i} = \alpha_i$,

and let $F^\bullet Rep_\alpha \subseteq Rep_\alpha$ preserve $F^\bullet$. Let $\mathbb{G}_\alpha := \prod_{i \in Q_0} GL(V^{\alpha_i})$ and

$\mathbb{P}_\alpha := \prod_{i\in Q_0} P_{\alpha_i}$, which fixes $F^\bullet$, where $P_{\alpha_i} \subseteq GL(V^{\alpha_i})$ for all i.

Let $\widetilde{Rep}_\alpha := \{(W, F^\bullet Rep_\alpha) \in Rep_\alpha \times \mathbb{G}_\alpha/\mathbb{P}_\alpha : W \in F^\bullet Rep_\alpha\}$ be the *generalized Grothendieck-Springer space of pairs*. Then

$$\mathbb{G}_\alpha \times_{\mathbb{P}_\alpha} F^\bullet Rep_\alpha \cong \widetilde{Rep}_\alpha, \quad \widetilde{Rep}_\alpha/\mathbb{G}_\alpha \cong F^\bullet Rep_\alpha/\mathbb{P}_\alpha$$

as orbit spaces.

Pathways ([Im14]). Definitions:

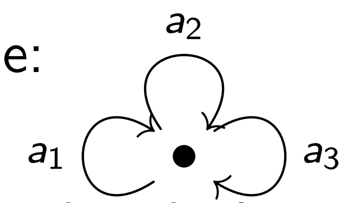
The *path algebra* $\mathbb{C}Q$ of Q is the $\mathbb{C}$-algebra with basis the paths in Q, with the product of two paths p and q given by $p \circ q = pq$ if $\mathrm{tail}(p) = \mathrm{head}(q)$; otherwise, $p \circ q = 0$.

A *relation* of a quiver Q is a subspace of $\mathbb{C}Q$ spanned by linear combinations of paths having a common source and a common target, and of length at least 2.

A *quiver with relations* is a pair (Q, I), where Q is a quiver and I is a two-sided ideal of $\mathbb{C}Q$ generated by relations.

A *quotient algebra* $\mathbb{C}Q/I$ is the path algebra of (Q, I).

Example:

Let Q: 

. Then $\mathbb{C}Q = \mathbb{C}\langle a_1, a_2, a_3\rangle$.
Let I be the ideal generated by $a_i a_j - a_j a_i$, $1 \leq i < j \leq 3$.
Then $\mathbb{C}Q/I = \mathbb{C}[a_1, a_2, a_3]$.

A path p is *reduced* if $[p] \neq 0$ in $\mathbb{C}Q/\langle q^2 : q \in \mathbb{C}Q, \mathrm{len}(q) \geq 1\rangle$, where $\mathrm{len}(q)$ is the number of arrows in q.

A *pathway* from vertex i to vertex j is a reduced path from i to j. We define *pathways* of a quiver Q to be the set of all pathways from vertex i to vertex j, where $i, j \in Q_0$.

Theorem([Im14], Thm 1.1): let $F^\bullet$ be $\nu = (1, 1, \ldots, 1) \in (\mathbb{Z}^{Q_0}_{\geq 0})^n$. Then Q is a quiver with at most two distinct pathways between any two vertices if and only if $\mathbb{C}[F^\bullet Rep_\alpha]^{\mathbb{U}_\alpha} \cong \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$.

Corollary (-): if $F^\bullet$ corresponds to $\nu^i = (k_i, \ldots, k_i) \in \mathbb{Z}^{Q_0}_{\geq 0}$ for each i, then Q has at most two distinct pathways between any two vertices if and only if $\mathbb{C}[F^\bullet Rep_\alpha]^{\mathbb{U}_\alpha} \cong \mathbb{C}[\mathfrak{l}^{\oplus Q_1}]$.

Corollary (-): above Corollary holds for *ADE*-Dynkin and affine type quivers.

Quiver Hecke algebras ([KL09], [KL11], [CR08], [Rou12]): motivation

Theorem 5.7 in [Rou12] and Theorem 3.6 [VV11] (general idea): connect quiver flag varieties and quiver Hecke algebras by proving that quiver Hecke algebras are Ext-algebras of certain sums of shifted simple perverse sheaves on quiver varieties.

Describe the structure of canonical (natural) bases of quantum groups (quantized Kac-Moody algebras) and their duals via

- generators and relations,
- embeddings,
- homomorphisms into "nicer" algebras.

Quiver Hecke algebras ([KL09], [KL11], [CR08], [Rou12]): motivation

Quantum affine categorify the upper half of a Lie group:

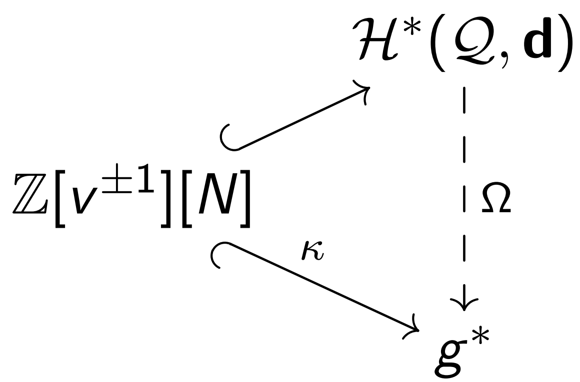
$$\mathbb{C}[N] \xrightarrow{\sim} \mathbb{C} \otimes K^0(\mathcal{C}'),$$

where $\mathcal{C}' \subseteq \mathcal{C}$ is a monoidal subcategory of the category of finite dimensional representations of quantum affine Kac-Moody algebras.

Quiver Hecke algebras categorify the lower half of quantum groups:

$$U_q^-(\mathfrak{g}) \supseteq {}_{\mathcal{A}}f \xrightarrow{\sim} \bigoplus_{\nu \in \mathbb{Z}_{\geq 0}^{Q_0}} K^0(R(\nu)), \theta_{\mathbf{i}} = \theta_{i_1}^{(a_1)} \cdots \theta_{i_r}^{(a_r)} \mapsto [P_{\mathbf{i}}],$$

where $\mathbf{i} = i_{i_1}^{(a_1)} \cdots i_{i_r}^{(a_r)} \in \mathrm{Seq}(\nu)$ and ${}_{\mathcal{A}}f$ is a $\mathbb{Z}[q^{\pm 1}]$-subalgebra generated by all products of quantum divided powers $\theta_i^{(a)}$, $i \in Q_0, 0 \leq a \in \mathbb{N}$ ([Lus10]).



- ([Rou07], Thm 4.6) the map from the integral form of the upper half of the quantum group to the graded dual Hall-Ringel bialgebra $\mathcal{H}^*(\mathcal{Q}, \mathbf{d})$
- ([Rup14], Thm 6.10) $\mathbb{Z}[v^{\pm 1}][N] \overset{\kappa}{\hookrightarrow} g^*$, $x_i \mapsto (i)$
- ([Rup14], Thm 7.1) Ω is the quantum shuffle character
- ([VV11], pg 11) $\bigoplus_{\nu \in \mathbb{Z}_{\geq 0}^{Q_0}} K^0(R(\nu)) \overset{\mathrm{ch}_v}{\hookrightarrow} g^*$, $[M] \mapsto \sum_{\mathbf{j} \in \mathbf{W}} \dim_v M_{\mathbf{j}} \cdot \mathbf{j}$

Note: $\mathrm{im}\, \mathrm{ch}_v = \mathbb{Z}[v^{\pm 1}][N]$. Problem: relate ch_v and Ω for all Q.
Partial result: [LV11] and [VV11] for symmetric Cartan matrices.

Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2010, Reprint of the 1997 edition.

Joseph Chuang and Raphaël Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification*, Ann. of Math. (2) **167** (2008), no. 1, 245–298.

Mee Seong Im, *Semi-invariants of filtered quiver representations with at most two pathways*, http://arxiv.org/pdf/1409.0702.pdf, 2014.

Mikhail Khovanov and Aaron D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory **13** (2009), 309–347.

______, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2685–2700.

George Lusztig, *Introduction to quantum groups*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010, Reprint of the 1994 edition.

Aaron D. Lauda and Monica Vazirani, *Crystals from categorified quantum groups*, Adv. Math. **228** (2011), no. 2, 803–861.

Thomas Nevins, *Stability and Hamiltonian reduction for Grothendieck-Springer resolutions*, www.math.uiuc.edu/~nevins/papers/b-hamiltonian-reduction-2011-0316.pdf, 2011.

Raphael Rouquier, *Hall algebras*, http://people.maths.ox.ac.uk/rouquier/hall7.pdf, 2007.

Raphaël Rouquier, *Quiver Hecke algebras and 2-Lie algebras*, Algebra Colloq. **19** (2012), no. 2, 359–410.

Dylan Rupel, *The Feigin Tetrahedron*, http://arxiv.org/abs/1401.4338, 2014.

M. Varagnolo and E. Vasserot, *Canonical bases and KLR-algebras*, J. Reine Angew. Math. **659** (2011), 67–100.

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Thank you.