Applications of filtered quiver varieties in representation theory

Mee Seong Im

December 13, 2013
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Applications of invariant theory

Applications

- David Mumford’s geometric invariant theory and Hilbert schemes in algebraic geometry
- Nakajima’s quiver variety and Gan and Ginzburg’s almost-commuting variety in representation theory
- Hamiltonian reduction construction in symplectic geometry
- Weight enumerators and self-dual codes in coding theory
- Grothendieck-Springer resolution in geometric representation theory
Finite group actions and classical problems I

▶ Consider $\mathbb{C}[x, y]$.
  ▶ Problem: find all $f \in \mathbb{C}[x, y]$ invariant under the linear transformation $f : x \to -x, y \to -y$.
    ▶ Invariant polynomials: $x^2, y^2, xy$.
    ▶ $S_2$-invariant subring:
      \[
      \mathbb{C}[x, y]^{S_2} = \mathbb{C}[x^2, y^2, xy] \cong \frac{\mathbb{C}[X, Y, Z]}{(XY - Z^2)}
      \]
  ▶ Problem: find all $f \in \mathbb{C}[x, y]$ invariant under the linear transformation $f : x \to -x, y \to y$.
    ▶ Invariant polynomials: $x^2, y$.
    ▶ $S_2$-invariant subring:
      \[
      \mathbb{C}[x, y]^{S_2} \cong \mathbb{C}[x^2, y].
      \]
  ▶ Definition: Let $G$ be a finite group and $V$ be a vector space. A (linear) representation of $G$ on $V$ is a group homomorphism $\rho : G \to GL(V)$. 

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Finite group actions and classical problems II

- If \( \rho \) is a representation of \( G \), then \( (g, v) \mapsto \rho(g)v \) is an **action** of \( G \) on \( V \).

- Definition: A vector space with a \( G \)-action by linear maps is also called a **\( G \)-module**.

- Definition: given a representation of a group \( G \) on a vector space \( V \), a regular (polynomial) function \( f \in \mathbb{C}[V] \) is called **\( G \)-invariant** or **invariant** if \( f(g.v) = f(v) \) for all \( g \in G \) and \( v \in V \).

- We denote \( \mathbb{C}[V]^G \subseteq \mathbb{C}[V] \) to be the subalgebra of invariant functions.
What is the geometry behind the (sub)algebra? I

▶ In the example when $\mathbb{C}[x, y]^{S_2} \cong \mathbb{C}[X, Y, Z]/\langle XY - Z^2 \rangle$, the corresponding variety is the cone

$$\text{Spec}(\mathbb{C}[x, y]^{S_2}) \cong \mathbb{C}^2 / S_2.$$ 

▶ In the example when $\mathbb{C}[x, y]^{S_2} \cong \mathbb{C}[x^2, y]$,

$$\text{Spec}(\mathbb{C}[x, y]^{S_2}) \cong \mathbb{C}^2 / S_2.$$
What is the geometry behind the (sub)algebra? II

- In an orbit space, points that lie in the same $G$-orbit are identified in order to obtain the variety corresponding to the invariant subring.
- Each point in $V/G$ corresponds to a distinct coset.
- What if $G$ is not finite?
  - For $\mathbb{C}^*$ acting on $\mathbb{C}^2$ via $\lambda.(x, y) = (\lambda x, \lambda^{-1}y)$, we have $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}[xy]$. But $\mathbb{C}^2/\mathbb{C}^*$ is not Hausdorff.
  - For $\mathbb{C}^*$ acting on $\mathbb{C}^2$ via $\lambda.(x, y) = (\lambda x, \lambda y)$, we have $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}$. Again, the orbit space $\mathbb{C}^2/\mathbb{C}^*$ is not Hausdorff. How should we make it $T_2$?
    - The origin lies in every orbit closure.
    - Any morphism constant on orbits is constant.
    - Topology on $\mathbb{C}^2/\mathbb{C}^*$ is undesirable.
    - A good candidate for $\mathbb{C}^2/\mathbb{C}^*$ is a point.
- Define $\mathbb{C}^2/\mathbb{C}^* := \text{Spec}(\mathbb{C}[x, y]^{\mathbb{C}^*}) =$ space of closed orbits.
Applications of filtered quiver varieties in representation theory

What is invariant theory?

Geometric perspective

Geometric invariant theory

- Definition: $\text{Spec}(R) :=$ the set of all proper prime ideals of $R$.
- Definition: given a character $\chi$ of $G$, define

\[
\mathbb{C}[V]^G,\chi := \{ f \in \mathbb{C}[V] : f(g \cdot v) = \chi(g)f(v) \ \forall g \in G, v \in V \}.
\]

- Definition: define $V//_{\chi} G := \text{Proj}(\bigoplus_{i \geq 0} \mathbb{C}[V]^G,\chi^i)$.
- Definition: given $S = \bigoplus_{i \geq 0} S_i$, we define $\text{Proj}(S) :=$ the set of all homogeneous prime ideals that do not contain the irrelevant ideal $S_+ = \bigoplus_{i > 0} S_i$.
- Use GIT techniques to construct new varieties and relate to classical problems.
Classical problems: let $G = GL_n(\mathbb{C})$.

- Suppose $G$ acts on $M_n$ via conjugation.
  - Problem: what is the geometry, algebra, and intuition behind the conjugation action?
  - The conjugation action is a change-of-basis.
  - From linear algebra, any square matrix can be put into Jordan canonical form.
  - $\mathbb{C}[M_n]^G \cong \mathbb{C}[\mathfrak{h}]^{S_n} \cong \mathbb{C}[\mathbb{C}^n]$.

- Suppose $G$ acts on $M_n^{\oplus k}$ via simultaneous conjugation: for $g \in G$ and $(A_1, \ldots, A_k)$, we have

$$g.(A_1, \ldots, A_k) = (gA_1g^{-1}, \ldots, gA_kg^{-1}).$$

Then $\mathbb{C}[M_n^{\oplus k}]^G \cong \mathbb{C}[\text{tr}(\text{closed quiver paths})]$. 
Classical problems: let $G = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$.

- Suppose $G$ acts on $M_n$ via the left-right action: for $(g, h) \in G$ and $A \in M_n$, we have $(g, h).A = gAh^{-1}$.
  Then $\mathbb{C}[M_n]^G \cong \mathbb{C}$ via 1-parameter subgroup techniques.

- Suppose $G$ acts on $M_n^{\oplus k}$ via the simultaneous left-right action: for $(g, h)$ and $(A_1, \ldots, A_k)$, we have

\[
(g, h). (A_1, \ldots, A_k) = (gA_1h^{-1}, \ldots, gA_kh^{-1}).
\]

Then $\mathbb{C}[M_n^{\oplus k}]^G \cong \mathbb{C}$ by elementary algebra or 1-parameter subgroup techniques.
Generalizing classical problems

- One way is via quiver representations.
- Definition: a quiver is a finite directed graph and a representation of a quiver assigns a finite dimensional vector space to each vertex and a linear map to each arrow.
- Denote $\text{Rep}(Q, \beta)$ as the quiver representation space. A product of $G := \prod_{i \in Q_0} GL_{\beta_i}(\mathbb{C})$ acts on a quiver representation as a change-of-basis.
- Examples:

\[ Q : \bullet \rightarrow \bullet \leftarrow \bullet \]

\[ Q' : \circ \rightarrow \bullet \rightarrow \bullet \]

- Suppose the dimension vector $\beta = (1, n)$ is assigned to $Q'$. This is the birth of a new and important area in mathematics.
Hamiltonian reduction in symplectic geometry

Suppose $G := GL_n(\mathbb{C})$ acts on $M_n \times \mathbb{C}^n$. Differentiate the above action: $\frac{d}{dt}(\exp(tv) \cdot (r, i))|_{t=0} = ([v, r], vi)$. Dualize $a$ to get $T^*(gl_n \times \mathbb{C}^n) \xrightarrow{\mu} g^*, (r, s, i, j) \mapsto [r, s] + ij$. Above construction is used to construct an almost-commuting variety.

**Theorem** (Crawley-Boevey, Gan-Ginzburg): $\mu^{-1}(0)$ is a complete intersection with $n + 1$ irreducible components.
Hamiltonian reduction in symplectic geometry

\[ Q' : \circ \rightarrow \bullet \]

**Theorem (Nakijima):**
\[ (\mathbb{C}^2)[n] \cong \mu^{-1}(0) \big/ \det G \rightarrow \cdots \rightarrow \mu^{-1}(0) \big/ \det^{-1} G \]

\[ \mu^{-1}(0) \big/ G \cong S^n(\mathbb{C}^2) \]

**Theorem (Derksen-Weyman):** given any acyclic \( Q \),
\[ \mathbb{C}[Rep(Q, \beta)] \prod_{i \in Q_0} SL_{n_i}(\mathbb{C}) \] is finitely generated.

**Theorem (Domokos-Zubkov, Schofield-Van den Bergh):**
given any \( Q \), \[ \mathbb{C}[Rep(Q, \beta)] \prod_{i \in Q_0} SL_{n_i}(\mathbb{C}) \] is finitely generated.
Interesting quiver varieties

- The study of the Jordan quiver reps $\bullet \circlearrowleft$ is equivalent to the study of the space of matrices under conjugation.
- The study of the Kronecker quiver reps $\bullet \rightarrow \bullet$ is equivalent to the study of the space of matrices under the left-right action.
- $ADE$-Dynkin quiver varieties
- Nakajima, Lusztig, and Calogero-Moser quiver varieties
- Restrict to a subgroup $\mathbb{H}$ of $G$ or restrict to a subspace $\text{Rep}(Q, \beta)$ of the vector space $\text{Rep}(Q, \beta)$
- Filtered representations of quiver varieties (Im)
  - Results
  - Open problems
Filtered quiver varieties

Definition: let \( Q \) be a quiver and let \( \beta \in \mathbb{Z}^{Q_0} \). Let \( F^\bullet \) be a filtration of vector spaces at each vertex \( i \in Q_0 \). We define \( F^\bullet \text{Rep}(Q, \beta) \subseteq \text{Rep}(Q, \beta) \) to be the subspace of all maps that preserve the filtration of vector spaces.

The product \( \mathbb{P} \subseteq \mathbb{G} \) of parabolic invertible matrices acts on \( F^\bullet \text{Rep}(Q, \beta) \) as a change-of-basis.

Problem: given any \( Q, \beta, \) and \( F^\bullet \), what is \( \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^\mathbb{U} \) where \( \mathbb{U} \) is the maximal unipotent subgroup of \( \mathbb{P} \)?

Example: let \( Q \) be the framed Jordan quiver \( \circ \rightarrow \bullet \circ \rightarrow \bullet \) and let \( \beta = (1, n) \). Let \( F^\bullet \) be the complete standard filtration of vector spaces at the nonframed vertex. Then \( F^\bullet \text{Rep}(Q, \beta) \cong b \times \mathbb{C}^n \) and \( B \leq GL_n(\mathbb{C}) \) acts on \( b \times \mathbb{C}^n \) via the adjoint action. What are \( b \times \mathbb{C}^n \parallel_{\chi} B \) for various \( \chi \)?
Basic assumption: let $F^\bullet$ be the complete standard filtration of vector spaces at each vertex of a quiver.

**Theorem (-)**: for a quiver of a finite Dynkin type with $\beta = (n, \ldots, n)$ and $F^\bullet$, $\mathbb{C}[b^{\oplus r}]^U \cong \mathbb{C}[t^{\oplus r}]$.

**Theorem (-)**: for an affine $\widetilde{A}_r$ quiver with a framing and $\beta = (n, \ldots, n, m)$ with $F^\bullet$, the subring $\mathbb{C}[b^{\oplus r+1} \oplus M_{n \times m}]^U$ is finitely generated.

**Corollary (-)**: for an affine $\widetilde{A}_r$ quiver with no framing and $\beta = (n, \ldots, n)$ with $F^\bullet$, $\mathbb{C}[b^{\oplus r+1}]^U \cong \mathbb{C}[t^{\oplus r+1}]$.

**Corollary (-)**: for an affine $\widetilde{D}_r$ quiver with no framing and $\beta = (n, \ldots, n)$ with $F^\bullet$, $\mathbb{C}[b^{\oplus r}]^U \cong \mathbb{C}[t^{\oplus r}]$. 

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Open problems

- Describe generators (and relations) of the following invariant rings.
  - Consider the 3-Kronecker quiver
    \[ \bullet \rightarrow \rightarrow \rightarrow \rightarrow \bullet \]
    with \( \beta = (n, n) \) with the complete standard filtration of vector spaces at each vertex. The subring of interest is \( \mathbb{C}[b^{\oplus 3}] U_n \times U_n \).
  - Consider the 2-Jordan quiver
    \[ \circ \rightarrow \bullet \rightarrow \circ \]
    with dimension vector \( n \) with the complete standard filtration of vector spaces. The subring of interest is \( \mathbb{C}[b^{\oplus 2}] U_n \).

- Together with the above two problems and results by Im, we then will be able describe the invariant ring for the filtered space for any quiver \( Q \).
Coding theory I

- Consider a noisy telegraph line from Chicago to NY which transmits 0’s and 1’s.
- Occasionally, 0 is received as a 1 and 1 is received as a 0.
- Problem: send many important messages down this line, quickly and reliably.
- Solution: send only a certain strings of 0’s and 1’s, which are called **codewords**.
- Example:

  ![Diagram]

  - Suppose 01010 is received.
  - The message is, more likely, 00000 (2 errors occurred), rather than 11111 (3 errors occurred).
Coding theory II

- Definition: The Hamming distance $\text{dist}(u, v)$ between two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ is the number of coordinates where $u_i \neq v_i$.

- Examples: $\text{dist}(01010, 00000) = 2$, $\text{dist}(0122, 2001) = 4$.

- Receiver should decode the received vector as the closest codeword, measured in Hamming distance.

- In general, if $d = \min$ Hamming distance between $u$ and $v$, the code can correct $e = \left\lfloor \frac{1}{2}(d - 1) \right\rfloor$ errors, where $\lfloor x \rfloor$ is the greatest integer not exceeding $x$.

- Definition: An $[n, k, d]$ code over $\mathbb{F}_q$ consists of $q^k$ codewords $(u_1, \ldots, u_n)$ which have Hamming distance at least $d$ apart and form a linear space.

- Definition: $n = \text{length}$, $k = \text{dimension}$, $d = \text{minimum distance}$ of the code.
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Coding theory III

Definition: Let $C$ be an $[n, k, d]$ code over $\mathbb{F}_q$. We define the **dual code** to be

$$C^\perp = \{(u_1, \ldots, u_n) : u \cdot v = \sum_{i=1}^{n} u_i v_i = 0 \ \forall v = (v_1, \ldots, v_n) \in C\}.$$ 

Definition: A **self-dual code** is one for which $C^\perp = C$.

If $C = C^\perp$, then $k = \frac{1}{2} n$; so $n$ is even.

Definition: The **(Hamming) weight** $\text{wt}(u)$ of $u$ is the number of nonzero $u_i$.

Since for all $u, v \in C$, we have

$$\text{dist}(u, v) = \text{wt}(u - v) = \text{wt}(w)$$

for some $w \in C$, the minimum distance $d$ between codewords is equal to the smallest weight of any nonzero codeword.
Definition: If an \([n, k, d]\) code \(C\) contains \(A_i\) codewords of weight \(i\), then the **weight enumerator** of \(C\) is

\[
W_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i = \sum_{u \in C} x^{n - \text{wt}(u)} y^{\text{wt}(u)},
\]

where \(x\) and \(y\) are indeterminates.

The weight enumerator of an \([n, k, d]\) code is a polynomial which tells the number of codewords of each weight.

**Theorem** (F. Jessie MacWilliams): if \(C\) is an \([n, k, d]\) code over \(\mathbb{F}_q\) with dual code \(C^\perp\), then

\[
W_{C^\perp}(x, y) = \frac{1}{q^k} W_C(x + (q-1)y, x - y).
\]

If \(C\) is self-dual, then \(k = n/2\) and

\[
W_C(x, y) = W_C \left( \frac{x + (q-1)y}{q^{1/2}}, \frac{x - y}{q^{1/2}} \right).
\]

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The end

Thank you. Questions?