Applications of filtered quiver varieties in representation theory

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December 13, 2013

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Applications

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Finite group actions and classical problems I

- Consider $\mathbb{C}[x, y]$.
  - Problem: find all $f \in \mathbb{C}[x, y]$ invariant under the linear transformation $f : x \to -x, y \to -y$.
    - Invariant polynomials: x^2, y^2, xy.
    - S_2-invariant subring:

$$\mathbb{C}[x, y]^{S_2} = \mathbb{C}[x^2, y^2, xy] \cong \frac{\mathbb{C}[X, Y, Z]}{\langle XY - Z^2 \rangle}$$

  - Problem: find all $f \in \mathbb{C}[x, y]$ invariant under the linear transformation $f : x \to -x, y \to y$.
    - Invariant polynomials: x^2, y.
    - S_2-invariant subring:

$$\mathbb{C}[x, y]^{S_2} \cong \mathbb{C}[x^2, y].$$

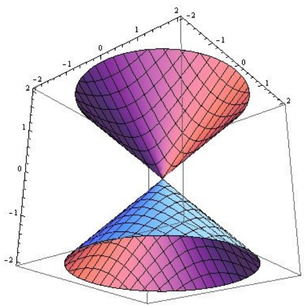
- Definition: Let G be a finite group and V be a vector space. A (**linear**) **representation** of G on V is a group homomorphism $\rho : G \to GL(V)$.

Finite group actions and classical problems II

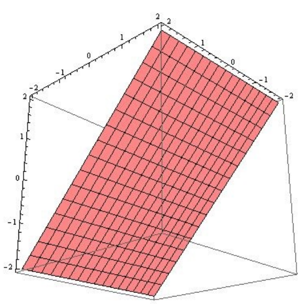
- If ρ is a representation of G, then $(g, v) \mapsto \rho(g)v$ is an **action** of G on V.
- Definition: A vector space with a G-action by linear maps is also called a G-**module**.
- Definition: given a representation of a group G on a vector space V, a regular (polynomial) function $f \in \mathbb{C}[V]$ is called G-**invariant** or **invariant** if $f(g.v) = f(v)$ for all $g \in G$ and $v \in V$.
- We denote $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$ to be the subalgebra of invariant functions.

What is the geometry behind the (sub)algbra? I

- In the example when $\mathbb{C}[x,y]^{S_2} \cong \mathbb{C}[X,Y,Z]/\langle XY - Z^2 \rangle$, the corresponding variety is the cone

$$\operatorname{Spec}(\mathbb{C}[x,y]^{S_2}) \cong$$

$$\cong \mathbb{C}^2/S_2.$$

- In the example when $\mathbb{C}[x,y]^{S_2} \cong \mathbb{C}[x^2,y]$,

$$\operatorname{Spec}(\mathbb{C}[x,y]^{S_2}) \cong$$

$$\cong \mathbb{C}^2/S_2.$$

What is the geometry behind the (sub)algebra? II

- In an orbit space, points that lie in the same G-orbit are identified in order to obtain the variety corresponding to the invariant subring.
- Each point in V/G corresponds to a distinct coset.
- What if G is not finite?
  - For $\mathbb{C}^*$ acting on $\mathbb{C}^2$ via $\lambda.(x, y) = (\lambda x, \lambda^{-1} y)$, we have $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}[xy]$. But $\mathbb{C}^2/\mathbb{C}^*$ is not Hausdorff.
  - For $\mathbb{C}^*$ acting on $\mathbb{C}^2$ via $\lambda.(x, y) = (\lambda x, \lambda y)$, we have $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}$. Again, the orbit space $\mathbb{C}^2/\mathbb{C}^*$ is not Hausdorff. How should we make it T_2?
    - The origin lies in every orbit closure.
    - Any morphism constant on orbits is constant.
    - Topology on $\mathbb{C}^2/\mathbb{C}^*$ is undesirable.
    - A good candidate for $\mathbb{C}^2/\mathbb{C}^*$ is a point.
  - Define $\mathbb{C}^2/\mathbb{C}^* := \mathrm{Spec}(\mathbb{C}[x, y]^{\mathbb{C}^*})$ = space of closed orbits.

Geometric invariant theory

- Definition: $\mathrm{Spec}(R) :=$ the set of all proper prime ideals of R.
- Definition: given a G-space V, define $V /\!\!/ G := \mathrm{Spec}(\mathbb{C}[V]^G)$.
- Definition: given a character χ of G, define

$$\mathbb{C}[V]^{G,\chi} := \{f \in \mathbb{C}[V] : f(g.v) = \chi(g)f(v) \ \forall g \in G, v \in V\}.$$

- Definition: define $V /\!\!/_{\chi} G := \mathrm{Proj}(\bigoplus_{i \geq 0} \mathbb{C}[V]^{G,\chi^i})$.
- Definition: given $S = \bigoplus_{i \geq 0} S_i$, we define $\mathrm{Proj}(S) :=$ the set of all homogeneous prime ideals that do not contain the irrelevant ideal $S_+ = \bigoplus_{i > 0} S_i$.
- Use GIT techniques to construct new varieties and relate to classical problems.

Classical problems: let $G = GL_n(\mathbb{C})$.

- Suppose G acts on M_n via conjugation.
  - Problem: what is the geometry, algebra, and intuition behind the conjugation action?
  - The conjugation action is a change-of-basis.
  - From linear algebra, any square matrix can be put into Jordan canonical form.
  - $\mathbb{C}[M_n]^G \cong \mathbb{C}[\mathfrak{h}]^{S_n} \cong \mathbb{C}[\mathbb{C}^n]$.
- Suppose G acts on $M_n^{\oplus k}$ via simultaneous conjugation: for $g \in G$ and $(A_1, \ldots, A_k)$, we have

$$g.(A_1, \ldots, A_k) = (gA_1g^{-1}, \ldots, gA_kg^{-1}).$$

  Then $\mathbb{C}[M_n^{\oplus k}]^G \cong \mathbb{C}[\mathrm{tr}(\text{closed quiver paths})]$.

Classical problems: let $G = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$.

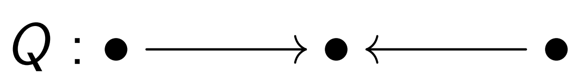
- Suppose G acts on M_n via the left-right action: for $(g, h) \in G$ and $A \in M_n$, we have $(g, h).A = gAh^{-1}$.
  Then $\mathbb{C}[M_n]^G \cong \mathbb{C}$ via 1-parameter subgroup techniques.
- Suppose G acts on $M_n^{\oplus k}$ via the simultaneous left-right action: for (g, h) and $(A_1, \ldots, A_k)$, we have

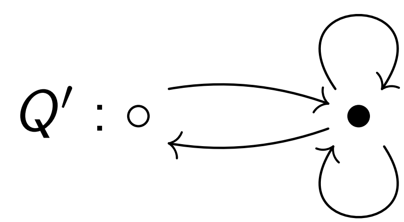
$$(g, h).(A_1, \ldots, A_k) = (gA_1h^{-1}, \ldots, gA_kh^{-1}).$$

  Then $\mathbb{C}[M_n^{\oplus k}]^G \cong \mathbb{C}$ by elementary algebra or 1-parameter subgroup techniques.

Generalizing classical problems

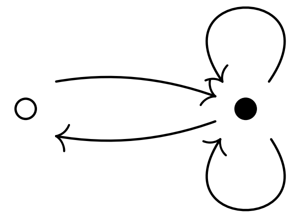
- One way is via quiver representations.
- Definition: a **quiver** is a finite directed graph and a **representation** of a quiver assigns a finite dimensional vector space to each vertex and a linear map to each arrow.
- Denote $Rep(Q, \beta)$ as the quiver representation space. A product of $\mathbb{G} := \prod_{i \in Q_0} GL_{\beta_i}(\mathbb{C})$ acts on a quiver representation as a change-of-basis.
- Examples:





- Suppose the dimension vector $\beta = (1, n)$ is assigned to Q'. This is the birth of a new and important area in mathematics.

Hamiltonian reduction in symplectic geometry

Q' :


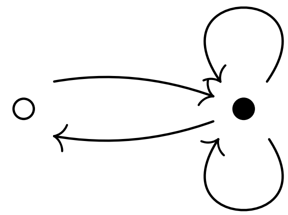
- Suppose $G := GL_n(\mathbb{C})$ acts on $M_n \times \mathbb{C}^n$.
- Differentiate the above action: $\mathfrak{g} \xrightarrow{a} \text{Vect}(\mathfrak{gl}_n \times \mathbb{C}^n)$,

$$a(v)(r,i) = \frac{d}{dt}(\exp(tv).(r,i))|_{t=0} = ([v,r], vi).$$

- Dualize a to get $T^*(\mathfrak{gl}_n \times \mathbb{C}^n) \xrightarrow{\mu} \mathfrak{g}^*$, $(r,s,i,j) \mapsto [r,s] + ij$.
- Above construction is used to construct an almost-commuting variety.

Theorem (Crawley-Boevey, Gan-Ginzburg): $\mu^{-1}(0)$ is a complete intersection with $n+1$ irreducible components.

Hamiltonian reduction in symplectic geometry

Q' :


Theorem (Nakijima):

$$(\mathbb{C}^2)^{[n]} \cong \mu^{-1}(0)/\!\!/_{\det} G \overset{\cong}{\dashrightarrow} \mu^{-1}(0)/\!\!/_{\det^{-1}} G$$

$$\mu^{-1}(0)/\!\!/_{\det} G \longrightarrow \mu^{-1}(0)/\!\!/ G \cong S^n(\mathbb{C}^2) \longleftarrow \mu^{-1}(0)/\!\!/_{\det^{-1}} G$$

Theorem (Derksen-Weyman): given any acyclic Q, $\mathbb{C}[Rep(Q,\beta)]^{\prod_{i\in Q_0} SL_{n_i}(\mathbb{C})}$ is finitely generated.

Theorem (Domokos-Zubkov, Schofield-Van den Bergh): given any Q, $\mathbb{C}[Rep(Q,\beta)]^{\prod_{i\in Q_0} SL_{n_i}(\mathbb{C})}$ is finitely generated.

Interesting quiver varieties

- The study of the Jordan quiver reps $\bullet \circlearrowleft$ is equivalent to the study of the space of matrices under conjugation.
- The study of the Kronecker quiver reps $\bullet \longrightarrow \bullet$ is equivalent to the study of the space of matrices under the left-right action.
- *ADE*-Dynkin quiver varieties
- Nakajima, Lusztig, and Calogero-Moser quiver varieties
- Restrict to a subgroup $\mathbb{H}$ of $\mathbb{G}$ or restrict to a subspace $\underline{Rep(Q, \beta)}$ of the vector space $Rep(Q, \beta)$
- Filtered representations of quiver varieties (Im)
  - Results
  - Open problems

Filtered quiver varieties

- Definition: let Q be a quiver and let $\beta \in \mathbb{Z}^{Q_0}$. Let $F^\bullet$ be a filtration of vector spaces at each vertex $i \in Q_0$. We define $F^\bullet Rep(Q, \beta) \subseteq Rep(Q, \beta)$ to be the subspace of all maps that preserve the filtration of vector spaces.
- The product $\mathbb{P} \subseteq \mathbb{G}$ of parabolic invertible matrices acts on $F^\bullet Rep(Q, \beta)$ as a change-of-basis.
- Problem: given any Q, β, and $F^\bullet$, what is $\mathbb{C}[F^\bullet Rep(Q, \beta)]^{\mathbb{U}}$ where $\mathbb{U}$ is the maximal unipotent subgroup of $\mathbb{P}$?
- Example: let Q be the framed Jordan quiver $\circ \longrightarrow \bullet \circlearrowleft$ and let $\beta = (1, n)$. Let $F^\bullet$ be the complete standard filtration of vector spaces at the nonframed vertex. Then $F^\bullet Rep(Q, \beta) \cong \mathfrak{b} \times \mathbb{C}^n$ and $B \leq GL_n(\mathbb{C})$ acts on $\mathfrak{b} \times \mathbb{C}^n$ via the adjoint action. What are $\mathfrak{b} \times \mathbb{C}^n /\!\!/_\chi B$ for various χ?

Basic assumption: let $F^\bullet$ be the complete standard filtration of vector spaces at each vertex of a quiver.

Theorem (-): for a quiver of a finite Dynkin type with $\beta = (n, \ldots, n)$ and $F^\bullet$, $\mathbb{C}[\mathfrak{b}^{\oplus r}]^{\mathbb{U}} \cong \mathbb{C}[\mathfrak{t}^{\oplus r}]$.

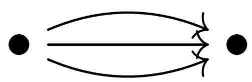
Theorem (-): for an affine $\widetilde{A}_r$ quiver with a framing and $\beta = (n, \ldots, n, m)$ with $F^\bullet$, the subring $\mathbb{C}[\mathfrak{b}^{\oplus r+1} \oplus M_{n\times m}]^{\mathbb{U}}$ is finitely generated.

Corollary (-): for an affine $\widetilde{A}_r$ quiver with no framing and $\beta = (n, \ldots, n)$ with $F^\bullet$, $\mathbb{C}[\mathfrak{b}^{\oplus r+1}]^{\mathbb{U}} \cong \mathbb{C}[\mathfrak{t}^{\oplus r+1}]$.

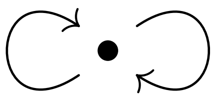
Corollary (-): for an affine $\widetilde{D}_r$ quiver with no framing and $\beta = (n, \ldots, n)$ with $F^\bullet$, $\mathbb{C}[\mathfrak{b}^{\oplus r}]^{\mathbb{U}} \cong \mathbb{C}[\mathfrak{t}^{\oplus r}]$.

Open problems

- Describe generators (and relations) of the following invariant rings.
  - Consider the 3-Kronecker quiver



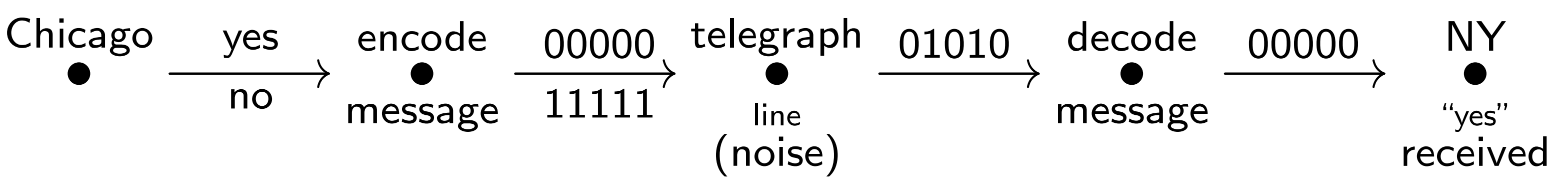
    with $\beta = (n, n)$ with the complete standard filtration of vector spaces at each vertex. The subring of interest is $\mathbb{C}[\mathfrak{b}^{\oplus 3}]^{U_n \times U_n}$.
  - Consider the 2-Jordan quiver



    with dimension vector n with the complete standard filtration of vector spaces. The subring of interest is $\mathbb{C}[\mathfrak{b}^{\oplus 2}]^{U_n}$.
- Together with the above two problems and results by Im, we then will be able describe the invariant ring for the filtered space for any quiver Q.

Coding theory I

- Consider a noisy telegraph line from Chicago to NY which transmits 0's and 1's.
- Occasionally, 0 is received as a 1 and 1 is received as a 0.
- Problem: send many important messages down this line, quickly and reliably.
- Solution: send only a certain strings of 0's and 1's, which are called **codewords**.
- Example:



  - Suppose 01010 is received.
  - The message is, more likely, 00000 (2 errors occurred), rather than 11111 (3 errors occurred).

Coding theory II

- Definition: The **Hamming distance** $\mathrm{dist}(\mathbf{u}, \mathbf{v})$ between two vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ is the number of coordinates where $u_i \neq v_i$.
- Examples: $\mathrm{dist}(01010, 00000) = 2$, $\mathrm{dist}(0122, 2001) = 4$.
- Receiver should decode the received vector as the closest codeword, measured in Hamming distance.
- In general, if $d =$ min Hamming distance between **u** and **v**, the code can correct $e = \left\lfloor \frac{1}{2}(d-1) \right\rfloor$ errors, where $\lfloor x \rfloor$ is the greatest integer not exceeding x.
- Definition: An $[n, k, d]$ **code** over $\mathbb{F}_q$ consists of q^k codewords $(u_1, \dots, u_n)$ which have Hamming distance at least d apart and form a linear space.
- Definition: $n =$ **length**, $k =$ **dimension**, $d =$ **minimum distance** of the code.

Coding theory III

- Definition: Let $\mathcal{C}$ be an $[n, k, d]$ code over $\mathbb{F}_q$. We define the **dual code** to be

$$\mathcal{C}^{\perp} = \{(u_1, \ldots, u_n) : \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = 0 \ \forall \mathbf{v} = (v_1, \ldots, v_n) \in \mathcal{C}\}.$$

- Definition: A **self-dual code** is one for which $\mathcal{C}^{\perp} = \mathcal{C}$.
- If $\mathcal{C} = \mathcal{C}^{\perp}$, then $k = \frac{1}{2}n$; so n is even.
- Definition: The (**Hamming**) **weight** $\mathrm{wt}\,(\mathbf{u})$ of $\mathbf{u}$ is the number of nonzero u_i.
- Since for all $\mathbf{u}, \mathbf{v} \in \mathcal{C}$, we have

$$\mathrm{dist}(\mathbf{u}, \mathbf{v}) = \mathrm{wt}\,(\mathbf{u} - \mathbf{v}) = \mathrm{wt}\,(\mathbf{w}) \text{ for some } \mathbf{w} \in \mathcal{C},$$

  the minimum distance d between codewords is equal to the smallest weight of any nonzero codeword.

Coding theory IV

- Definition: If an $[n, k, d]$ code $\mathcal{C}$ contains A_i codewords of weight i, then the **weight enumerator** of $\mathcal{C}$ is

$$W_{\mathcal{C}}(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i = \sum_{\mathbf{u}\in\mathcal{C}} x^{n-\mathrm{wt}\,(\mathbf{u})} y^{\mathrm{wt}\,(\mathbf{u})},$$

where x and y are indeterminates.
- The weight enumerator of an $[n, k, d]$ code is a polynomial which tells the number of codewords of each weight.

Theorem (F. Jessie MacWilliams): if $\mathcal{C}$ is an $[n, k, d]$ code over $\mathbb{F}_q$ with dual code $\mathcal{C}^\perp$, then

$$W_{\mathcal{C}^\perp}(x,y) = \frac{1}{q^k} W_{\mathcal{C}}(x+(q-1)y, x-y).$$

- If $\mathcal{C}$ is self-dual, then $k = n/2$ and

$$W_{\mathcal{C}}(x,y) = W_{\mathcal{C}}\left(\frac{x+(q-1)y}{q^{1/2}}, \frac{x-y}{q^{1/2}}\right).$$

The end

Thank you. Questions?