On semi-invariants of filtered representations of quivers and the cotangent bundle of the enhanced Grothendieck-Springer resolution

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Standing assumption: work over $\mathbb{C}$. Definitions:

A quiver $Q$ is a directed graph. Assume $Q$ to be finite, i.e., $Q$ has finite number $\{1, 2, \ldots, Q_0\}$ of vertices and finite number $\{a_1, \ldots, a_{Q_1}\}$ of arrows which come equipped with two functions:

for each arrow $i \xrightarrow{a} j$, $t, h : Q_1 \rightarrow Q_0$ map $t(a) = i$ and $h(a) = j$.

A representation of a quiver assigns a finite-dimensional vector space to each vertex and a linear map to each arrow.

A dimension vector of $Q$ is an element of the form $\beta \in \mathbb{Z}_{\geq 0}^{Q_0}$.

Given $Q$ and $\beta$, the representation space is

$$Rep(Q, \beta) := \prod_{a \in Q_1} \text{Hom}(\mathbb{C}^{\beta t(a)}, \mathbb{C}^{\beta h(a)}).$$

$Rep(Q, \beta)$ has a natural $G_\beta$-action, where $G_\beta := \prod_{i \in Q_0} \text{GL}_{\beta_i}$. 
Background. Examples:

- 2-Jordan $a_1 \quad 1 \quad a_2$, 3-Kronecker $1 \quad a_1 \quad a_2 \quad a_3 \quad 2$,

- Cycle $1 \quad a_1 \quad 2$, loop $1 \quad a$,

- $ADE$-Dynkin if the underlying graph is of $ADE$-Dynkin type,

- Affine (type) $\tilde{A}_r$-Dynkin

Relations of quiver representations to classical linear algebra: the study of $GL_n$-orbits on $\mathfrak{gl}_n = \text{Lie}(GL_n)$. 
Filtered quiver representations. Definitions:

Assume $Q$ and $\beta$ as before. Let

$$F_i^\bullet : \{0\} = U_i^0 \subseteq U_i^1 \subseteq U_i^2 \subseteq \ldots \subseteq U_i^N = \mathbb{C}^{\beta_i}$$

be a sequence of vector spaces, one for each $i \in Q_0$. Then $F^\bullet \text{Rep}(Q, \beta)$ is a subspace of $\text{Rep}(Q, \beta)$ whose linear maps preserve the filtration of vector spaces at every level. $F^\bullet \text{Rep}(Q, \beta)$ is called a filtered quiver variety.

Let $P_i \subseteq GL_{\beta_i}$ be the largest subgroup preserving the filtration of vector spaces at vertex $i$. Then $P_\beta := \prod_{i \in Q_0} P_i$ acts on $F^\bullet \text{Rep}(Q, \beta)$ as a change-of-basis.
Filtered quiver representations. Examples:

- $Q: \begin{array}{ccc} \bullet & \circlearrowleft & \bullet \end{array}, \beta = n$, $F^\bullet$ is the complete standard filtration of vector spaces. Then $F^\bullet \text{Rep}(Q, \beta) = \mathfrak{b}_n$, the space of $n \times n$ upper triangular matrices, with $B$-conjugation action on $\mathfrak{b}_n$, where $\text{Lie}(B) = \mathfrak{b}_n$.

- $Q: \begin{array}{ccc} \bullet & \rightarrow & \bullet \end{array}, \beta = (n, n)$, $F^\bullet$ is the complete standard filtration of vector spaces at each vertex. Then $F^\bullet \text{Rep}(Q, \beta) = \mathfrak{b}_n$ and $B_n \times B_n \subseteq GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acts on the filtered representation space via the left-right action.

Relations to classical ($19^{th}$ century) linear algebra: the study of $B$-orbits on $\mathfrak{b} = \text{Lie}(B)$. 
Grothendieck-Springer resolution.

There are embeddings $B \hookrightarrow GL_n$ and $b \hookrightarrow gl_n$ such that $\varphi$ is $\psi$-equivariant. Consider

$$\pi: GL_n \times_B b \rightarrow gl_n.$$ 

Lemma

Given $\tilde{gl}_n := \{(x, b) \in gl_n \times GL_n/B : x \in b\}$, $\tilde{gl}_n \cong GL_n \times_B b$.

Proof.

There is a map $(GL_n \times b)/B \rightarrow \tilde{gl}_n$, where $(g, x) \mapsto (g x g^{-1}, (g.B)/B)$. This map is $GL_n$-equivariant and is an isomorphism.
Motivation

Grothendieck-Springer resolution.

Since $GL_n$ acts on $\widetilde{\mathfrak{gl}}_n$ via $g.(x, b) = (xg^{-1}, gbg^{-1})$ and on $\mathfrak{gl}_n$ via the adjoint action, $\mathfrak{gl}_n \rightarrow \mathfrak{gl}_n$ is a $GL_n$-equivariant map.

Lemma

There is a bijection between $GL_n$-orbits on $\widetilde{\mathfrak{gl}}_n$ and $B$-orbits on $\mathfrak{b}$.

Proof.

Consider $(GL_n \times \mathfrak{b})/B \rightarrow \mathfrak{b}/B$, where $(g, x) \mapsto gxg^{-1}$. This map is well-defined up to the $B$-conjugation action. Since the map is $GL_n$-equivariant, it descends to an isomorphism $\widetilde{\mathfrak{gl}}_n/GL_n \cong \mathfrak{b}/B$ as orbit spaces.

Moral of the story: study the $B$-action on $\mathfrak{b}$.

Furthermore, $\mathbb{P}_\beta$-action on $F^\bullet \text{Rep}(Q, \beta)$ generalizes $B$-action on $\mathfrak{b}$ (21st century).
Comparing $\mathbb{U}_\beta$-invariants and $\mathbb{P}_\beta$-semi-invariants, where $\mathbb{U}_\beta$ is the maximal unipotent subgroup of $\mathbb{P}_\beta$.

Definition: $f \in \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{P}_\beta}$ is an invariant polynomial if $f(g.x) = f(x)$ for all $g \in \mathbb{P}_\beta$ and $x \in F^\bullet \text{Rep}(Q, \beta)$.

Definition: $f \in \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{P}_\beta, \chi}$ is a semi-invariant polynomial if $f(g.x) = \chi(g)f(x)$ for all $g \in \mathbb{P}_\beta$ and $x \in F^\bullet \text{Rep}(Q, \beta)$, where $\chi : \mathbb{P}_\beta \to \mathbb{C}^*$ is an algebraic group homomorphism, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The ring of semi-invariant polynomials is $\bigoplus_{\chi} \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{P}_\beta, \chi}$.

Semi-invariants under the $\mathbb{P}_\beta$-action are invariant for $\mathbb{U}_\beta$-action and $\mathbb{U}_\beta$-invariant polynomials that are homogeneous with respect to a generalized $Q_0$-grading are also semi-invariant (for some $\chi$) for the $\mathbb{P}_\beta$-action. Thus, $\bigoplus_{\chi} \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{P}_\beta, \chi} \cong \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{U}_\beta}$. 
\[ \mathbb{U}_\beta \text{-invariants and } \mathbb{P}_\beta \text{-semi-invariants: why are they interesting?} \]

Use invariant and semi-invariant polynomials to construct new and interesting varieties; that is,

- construct the **affine quotient**

\[
F^\bullet \text{Rep}(Q, \beta) / \mathbb{P}_\beta := \text{Spec}(\mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{P}_\beta})
\]

of the vector space \( F^\bullet \text{Rep}(Q, \beta) \) by \( \mathbb{P}_\beta \),

- construct the **geometric (GIT) quotient**

\[
F^\bullet \text{Rep}(Q, \beta) / \chi \mathbb{P}_\beta := \text{Proj}(\bigoplus_{i \geq 0} \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{\mathbb{P}_\beta, \chi^i})
\]

of the space \( F^\bullet \text{Rep}(Q, \beta) \) by \( \mathbb{P}_\beta \) twisted by \( \chi \).
Filtered quiver varieties of finite \( ADE \)-Dynkin type.

Basic assumption: let \( F^\bullet \) be the complete standard filtration of vector spaces at each vertex. Let \( t_n \subseteq \mathfrak{gl}_n \) be the set of complex diagonal matrices.

**Theorem (Im)**

If \( Q \) is an \( ADE \)-Dynkin quiver and \( \beta = (n, \ldots, n) \in \mathbb{Z}_{\geq 0}^Q \), then \( \mathbb{C}[F^\bullet \text{Rep}(Q, \beta)]^{U_\beta} \cong \mathbb{C}[t_n^{\oplus Q_1}] \).

Sketch of proof:

- It is clear that \( \mathbb{C}[t_n^{\oplus r-1}] \subseteq \mathbb{C}[b_n^{\oplus r-1}]^{U_\beta} \).
- Consider the equioriented finite \( A_r \)-quiver:

\[
\begin{array}{cccccccc}
1 & \xrightarrow{a_1} & 2 & \xrightarrow{a_2} & \cdots & r-2 & \xrightarrow{a_{r-2}} & r-1 & \xrightarrow{a_{r-1}} & r \\
\bullet & \rightarrow & \bullet & \rightarrow & \cdots & \bullet & \rightarrow & \bullet & \rightarrow & \bullet
\end{array}
\]

- Let \( A_\alpha = ((\alpha)_{a_{st}}) \) be a general representation of \( a_\alpha \).
Filtered quiver varieties of finite ADE-Dynkin type.

- Fix a total ordering $\leq$ on pairs $(i, j)$, where $1 \leq i \leq j \leq n$, by defining $(i, j) \leq (i', j')$ if either
  - $i < i'$ or
  - $i = i'$ and $j > j'$.
- For each $(i, j)$, we can write $f \in \mathbb{C}[b_n^{\oplus r-1}]^U_\beta$ as
  \[ f = \sum_K a^K f_{ij,K}, \text{ where } f_{ij,K} \in \mathbb{C}[\{(\alpha)a_{st} : (s, t) \neq (i, j)\}], \]
  where $a^K_{ij} := \prod_{\alpha=1}^{r-1} (\alpha)a_{ij}^{k\alpha}$.
- Fix the least pair (under $\leq$) $(i, j)$ with $i < j$ for which there exists $K \neq (0, \ldots, 0)$ with $f_{ij,K} \neq 0$; we will continue to denote it by $(i, j)$. If such least pair does not exist, we’re done.
Filtered quiver varieties of finite $ADE$-Dynkin type.

Let $K = (k_1, \ldots, k_{r-1})$. Let $m \geq 1$ be the least integer satisfying the following: for all $p < m$, if some component $k_p$ in $K$ is strictly greater than 0, then $f_{ij,K} = 0$.

Let $U_{ij}$ be the subgroup of matrices of the form $u_{ij} := (I_n, \ldots, I_n, \hat{u}_m, I_n, \ldots, I_n)$, where $I_n$ is the $n \times n$ identity matrix and $\hat{u}_m$ is the matrix with 1 along the diagonal, the variable $u$ in the $(i,j)$-entry, and 0 elsewhere.

\[
\begin{align*}
  u_{ij} \cdot (\alpha) a_{st} &= \begin{cases} 
    (m) a_{ij} + (m) a_{ii} u & \text{if } \alpha = m \text{ and } (s, t) = (i, j), \\
    (\alpha) a_{st} & \text{if } s > i \text{ or } s = i \text{ and } t < j.
  \end{cases}
\end{align*}
\]
Filtered quiver varieties of finite ADE-Dynkin type.

- Now write
  \[ f = \sum_{k \geq 0} (m) a_{ij}^k F_k, \]
  where \( F_k \in \mathbb{C}[\{(\alpha) a_{st} : (s, t) \geq (i, j) \text{ and if } (s, t) = (i, j), \text{ then } \alpha > m\}] =: R_0. \]

- We have
  \[ 0 = u_{ij}.f - f = \sum_{k \geq 1} \sum_{1 \leq l \leq k} (m)a_{ij}^{k-l} (m)a_{ii}^l u^l \binom{k}{l} F_k. \]

- \( \{(m)a_{ij}^{k-l} u^l : 1 \leq l \leq k, k \geq 0\} \) is linearly independent over \( R_0. \)

- Contradiction!
If $Q$ is an $ADE$-Dynkin quiver, then all classical semi-invariant techniques are applicable!

Classical techniques (for reductive groups) given by

- Schofield-van den Bergh (1999)
- Derksen-Weyman (2000)
- Domokos-Zubkov (2001)

are applicable. For more details, see Appendix of the slides.
Pathways. Definitions:

A nontrivial path is a sequence $a_m \cdots a_1$ of arrows such that $t(a_{i+1}) = h(a_i)$ for all $1 \leq i < m$. We write $e_i$ as the trivial (empty) path at vertex $i$.

The path algebra $\mathbb{C}Q$ of $Q$ is the $\mathbb{C}$-algebra with basis the paths in $Q$, with the product of two paths $p$ and $q$ given by $p \circ q = pq$ if $t(p) = h(q)$; otherwise, $p \circ q = 0$.

A relation of a quiver $Q$ is a subspace of $\mathbb{C}Q$ spanned by linear combinations of paths having a common source and a common target, and of length at least 2 (Michel Brion).

A quiver with relations is a pair $(Q, I)$, where $Q$ is a quiver and $I$ is a two-sided ideal of $\mathbb{C}Q$ generated by relations.

The quotient algebra $\mathbb{C}Q/I$ is the path algebra of $(Q, I)$.
Pathways.

Example:

Let $Q$: \[ a_1 \rightarrow \bullet \rightarrow a_2 \rightarrow a_3. \] Then $\mathbb{C}Q = \mathbb{C}\langle a_1, a_2, a_3 \rangle$.

Let $I$ be the ideal generated by $a_ia_j - a_ja_i$, $1 \leq i < j \leq 3$. Then $\mathbb{C}Q/I = \mathbb{C}[a_1, a_2, a_3]$.

Definition: a path $p$ is reduced if it is the class $[p] \neq 0$ in $\mathbb{C}Q/\langle q^2 : q \in \mathbb{C}Q, l(q) \geq 1 \rangle$, where $l(q)$ is the number of arrows in $q$.

Definition: a pathway from vertex $i$ to vertex $j$ is a reduced path from $i$ to $j$. We define pathways of a quiver $Q$ to be the set of all pathways from vertex $i$ to vertex $j$, where $i, j \in Q_0$.

More examples on the next slide.
Pathways. Examples:

- **Q:**  \[ \begin{array}{c}
\text{1} \\
\bullet \\
a \\
\end{array} \]
Paths of \( Q \) consist of \( e_1, a, a^2, a^3, \ldots \).
Pathways of \( Q \) consist of \( e_1 \) and \( a \).
This quiver has at most 2 pathways.

- **Q:**  \[ \begin{array}{c}
\text{1} \\
\bullet \\
a_1 \\
a_2 \\
a_3 \\
\bullet \\
\end{array} \]
There is one pathway from vertex 1 to vertex 1: \( e_1 \).
There are 3 pathways from vertex 1 to vertex 2: \( a_1, a_2, a_3 \).
There is one pathway from vertex 2 to vertex 2: \( e_2 \).
This quiver has at most 3 pathways between any two vertices.

- **Q:**  \[ \begin{array}{c}
\text{1} \\
\bullet \\
a_1 \\
a_2 \\
\end{array} \]
Pathways consist of \( e_1, a_1, a_2, a_1a_2, a_2a_1, a_1a_2a_1, a_2a_1a_2 \).
This \( Q \) has more than 2 pathways.
Examples of quivers with at most two pathways between any two vertices.

- ADE-Dynkin quivers,
- Framed $\tilde{A}_r$-Dynkin quivers, which includes $\circ \rightarrow \bullet \rightarrow \circ$,
- Star-shaped quivers have $k$ legs (each of length $s_k$), e.g.,

\[
\begin{array}{c}
[1,s_1] \\
\circ \rightarrow \bullet \leftarrow \bullet \quad \cdots \\
\end{array}
\quad
\begin{array}{c}
[1,2] \\
\bullet \rightarrow \circ \leftarrow \bullet \\
\end{array}
\quad
\begin{array}{c}
[1,1] \\
\circ \rightarrow \bullet \\
\end{array}
\]

\[
\begin{array}{c}
[2,s_2] \\
\circ \rightarrow \bullet \rightarrow \bullet \leftarrow \circ \quad \cdots \\
\end{array}
\quad
\begin{array}{c}
[2,2] \\
\bullet \rightarrow \circ \rightarrow \bullet \leftarrow \circ \\
\end{array}
\quad
\begin{array}{c}
[2,1] \\
\bullet \rightarrow \circ \leftarrow \circ \\
\end{array}
\quad
\begin{array}{c}
1 \\
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

\[
\begin{array}{c}
[k,s_k] \\
\bullet \leftarrow \bullet \leftarrow \bullet \quad \cdots \\
\end{array}
\quad
\begin{array}{c}
[k,2] \\
\bullet \rightarrow \circ \\
\end{array}
\quad
\begin{array}{c}
[k,1] \\
\circ \rightarrow \bullet \\
\end{array}
\]

\[\text{Mee Seong Im 19}\]
Examples of quivers with at most two pathways.

- Comet-shaped quivers have $k$ legs (each of length $s_k$), with 1 loop on the central vertex, e.g.,

\[
\begin{align*}
\text{[1,s1]} & \quad \bullet \quad \longrightarrow \quad \bullet \quad \leftarrow \quad \bullet \quad \cdots \quad \bullet \quad \leftarrow \quad \bullet \\
\text{[2,s2]} & \quad \bullet \quad \longrightarrow \quad \bullet \quad \longrightarrow \quad \bullet \quad \cdots \quad \bullet \quad \longrightarrow \quad \bullet \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\text{[k,s_k]} & \quad \bullet \quad \leftarrow \quad \bullet \quad \leftarrow \quad \bullet \quad \cdots \quad \bullet \quad \longrightarrow \quad \bullet
\end{align*}
\]
Examples of quivers with at most two pathways.

- A quiver whose underlying graph is

  ![Quiver Diagram]

  with the condition that vertices $2, 3, \ldots, k$ are a source or a sink (in any order).
Quivers with at most two pathways.

Basic assumption: let $F^\bullet$ be the complete standard filtration of vector spaces at each nonframed vertex. Let $t_n \subset gl_n$ be the set of complex diagonal matrices, and let $U_\beta := U^{Q_0} \subset B^{Q_0}$, $B \subset GL_n(\mathbb{C})$ is subgroup of invertible upper triangular matrices.

Theorem (Im)

Let $Q$ be a quiver and let $\beta = (n, \ldots, n)$. Then $Q$ is a nonframed quiver with at most two distinct pathways between any two vertices if and only if $\mathbb{C}[F^\bullet Rep(Q, \beta)]^{U_\beta} \cong \mathbb{C}[t^{\oplus Q_1}]$, where $Q_1$ is the set of arrows whose tail and head are nonframed vertices.

Remark: want to use classical techniques to obtain semi-invariants for certain filtered quiver representations? Then restrict to quivers with at most 2 pathways between any two of its vertices.
Quivers with at most two pathways.

Theorem (Im)

Let $Q$ be a quiver and let $\beta = (n, \ldots, n)$. Then $Q$ is a nonframed quiver with at most two distinct pathways between any two vertices if and only if $\mathbb{C}[F\mathcal{R}(Q, \beta)]^U_{\beta} \cong \mathbb{C}[t^{\oplus Q_1}]$, where $Q_1$ is the set of arrows whose tail and head are nonframed vertices.

That is, suppose $Q$ is a nonframed quiver. Then

$Q$ has at most two distinct pathways between any two vertices if and only if $\mathbb{C}[F\mathcal{R}(Q, \beta)]^U_{\beta} \cong \mathbb{C}[t^{\oplus Q_1}]$, where $Q_1$ is the number of arrows of $Q$.

Sketch of proof: similar to the proof for filtered quiver varieties for finite $ADE$-Dynkin type.
The end (unless there is more time).

Thank you. Questions?
Construction of $T^*(\mathfrak{b} \times \mathbb{C}^n)$, cf. Nevins’ manuscript.

Consider $B$-action on $\mathfrak{b} \times \mathbb{C}^n$ via $b.(r, i) = (brb^{-1}, bi)$ and $G \times B$-action on $G \times \mathfrak{b} \times \mathbb{C}^n$ via $g.(g', r, i) = (g'g^{-1}, r, gi)$ and $b.(g', r, i) = (g'b^{-1}, brb^{-1}, i)$, where $b \in B$ and $g \in G$.

This gives two moment maps

$$T^*(\mathfrak{b} \times \mathbb{C}^n) \xrightarrow{\mu_B} \mathfrak{b}^* \cong \mathfrak{gl}_n/\mathfrak{u}^+, (r, s, i, j) \mapsto [r, s] + ij,$$

and

$$T^*(G \times \mathfrak{b} \times \mathbb{C}^n) \xrightarrow{\mu_{G \times B}} \mathfrak{g}^* \times \mathfrak{b}^*, (g', \theta, r, s, i, j) \mapsto (\theta - ij, [r, s] + \bar{\theta}),$$

where $\bar{\theta} : \mathfrak{gl}_n \to \mathfrak{b}^*$.

There is a bijection of $B$-orbits on $\mu_B^{-1}(0)$ and $G \times B$-orbits on $\mu_{G \times B}^{-1}(0)$:

$$\mu_B^{-1}(0)/B \cong \mu_{G \times B}^{-1}(0)/G \times B \cong T^*(G \times \mathfrak{b} \times \mathbb{C}^n/G \times B) \cong T^*(\mathfrak{g} \times \mathbb{C}^n/G).$$
On semi-invariants of filtered representations of quivers and the cotangent bundle of the enhanced Grothendieck-Springer resolution

Cotangent bundle of enhanced Grothendieck-Springer resolution

Results

Results for $T^*(\mathfrak{b} \times \mathbb{C}^n)$.

We will thus study $\mu_B^{-1}(0)/B$.

Definition: an element of $\mathfrak{b}$ is regular if its stabilizer dimension is minimal. An element in $\mathfrak{b}$ is semisimple if it is diagonalizable.

Let $\mu_B^{-1}(0)^{rss}$ be the restriction of $\mu_B^{-1}(0)$ to the regular semisimple locus, i.e., it is the locus where eigenvalues of $\mathfrak{b}$ are pairwise distinct.

Proposition (Im)

$$\mu_B^{-1}(0)^{rss} \sslash B \cong \mathbb{C}^{2n} \setminus \Delta_n,$$

where

$$\Delta_n = \{(x_1, \ldots, x_n, 0, \ldots, 0) : x_i = x_j \forall i \neq j\}.$$
Open problems for the Hamiltonian reduction of the cotangent bundle of the enhanced Grothendieck-Springer resolution.

Nevins’ Conjecture: $\mu_B^{-1}(0)$ is a complete intersection.

Study

$$
\begin{align*}
\mu_B^{-1}(0) & \\
\mu_B^{-1}(0) & \psi_{\chi,\chi'} & \mu_B^{-1}(0)\\
\mu_B^{-1}(0) & \chi B & \psi_{\chi,\chi'} & \mu_B^{-1}(0) & \chi' B\\
\mu_B^{-1}(0) & / / & \chi B & \psi_{\chi,\chi'} & \mu_B^{-1}(0) & / / & \chi' B.
\end{align*}
$$

Why? What is the motivation?
Motivation for open problems.

Let $G := GL_n(\mathbb{C})$ act on $M_n \times \mathbb{C}^n$.

**Theorem (Crawley-Boevey, Gan-Ginzburg)**

$\mu^{-1}(0)$ is a complete intersection with $n + 1$ irreducible components.

**Theorem (Nakajima)**

$$\left(\mathbb{C}^2\right)^n \cong \mu^{-1}(0) \left/ \det G \right. \\
\mu^{-1}(0) \left/ \det^{-1} G \right. \cong S^n(\mathbb{C}^2)$$
The end.

Thank you. Questions?
If $Q$ is a quiver with at most 2 pathways, then all classical semi-invariant techniques are applicable!

Let $Q = (Q_0, Q_1)$ be an arbitrary quiver (where cycles are allowed) and let $\beta$ be a dimension vector. Choose a set $v_1, \ldots, v_n, w_1, \ldots, w_m \in Q_0$ of vertices (possibly repeating) such that

$$\sum_{i=1}^{n} \beta(v_i) = \sum_{j=1}^{m} \beta(w_j). \quad (1)$$

Let $W \in \text{Rep}(Q, \beta)$ be a general representation and consider

$$M : \bigoplus_{i=1}^{n} W(v_i) \longrightarrow \bigoplus_{j=1}^{m} W(w_j),$$

where $M = (m_{ij})$, with each $m_{ij}$ being a linear combination of a general representation of paths in $Q$ from $v_i$ to $w_j$, including the zero path which corresponds to the zero matrix and the identity matrix if $v_i = w_j$. 

**Theorem (Domokos-Zubkov)**

*Polynomial coefficients of the determinant of $M$ are in the algebra $\mathbb{C}[\text{Rep}(Q, \beta)]^{SL_\beta}$. Choose all possible combination of vertices satisfying (1) and all possible combination of representations of paths to obtain polynomial generators of $\mathbb{C}[\text{Rep}(Q, \beta)]^{SL_\beta}$.*

Take home message from this technique: works for any quiver!

Example

Suppose $Q$: $\begin{tikzpicture}[baseline=(current bounding box.center),scale=0.5]
\node (1) at (0,0) {$1$};
node (a) at (1,0) {$a$};
\node (v1) at (3,0) {$\leftarrow$};
\node (w1) at (2,0) {$\leftarrow$};
\path[->] (a) edge (1);
\end{tikzpicture}$ and $\beta = 2$.
Let $W(a) = (a_{ij})$ be a general representation of $\text{Rep}(Q, \beta)$.
Let $n = m = 1$ with $v_1 = w_1 = 1$.
Let $M : W(1) = \mathbb{C}^2 \longrightarrow W(1) = \mathbb{C}^2$, where $M = (tW(a))$.
Then $\det(M) = t^2 \det(W(a)) \Rightarrow \det(W(a))$ is an invariant.
Now let $n = m = 2$ with $v_1 = v_2 = w_1 = w_2 = 1$.
Then $M : \mathbb{C}^2 \oplus \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \oplus \mathbb{C}^2$, and let

$$M = \begin{pmatrix} sW(a) & tI_2 \\ uI_2 & vI_2 \end{pmatrix} = \begin{pmatrix} s & 0 & t & 0 \\ 0 & s & 0 & t \\ u & 0 & v & 0 \\ 0 & u & 0 & v \end{pmatrix},$$

where $s, t, u, v$ are formal variables.

Example (continued)

Then

\[
\det(M) = t^2 u^2 - (a_{11} + a_{22})stuv + (a_{11}a_{22} - a_{12}a_{21})s^2v^2,
\]

and \( \text{tr}(W(a)) = a_{11} + a_{22} \) is also an invariant polynomial. Thus, \( \mathbb{C}[\text{Rep}(Q, \beta)]^{SL_\beta} = \mathbb{C}[\text{tr}(W(a)), \det(W(a))] \).

Coincides with the classical result that generators of the ring of invariant polynomials are precisely the coefficients of the characteristic polynomial of \( W(a) \).