On filtered representations of quivers with at most two pathways and on the generalized Grothendieck-Springer resolution

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Standing assumption: work over $\mathbb{C}$. Definitions:

A *quiver* Q is a directed graph. Assume Q to be finite, i.e., Q has finite number $\{1, 2, \dots, Q_0\}$ of vertices and finite number $\{a_1, \dots, a_{Q_1}\}$ of arrows which come equipped with two functions: for each arrow $\overset{i}{\bullet} \xrightarrow{a} \overset{j}{\bullet}$, $t, h : Q_1 \to Q_0$ map $t(a) = i$ and $h(a) = j$.

A *representation* of a quiver assigns a finite-dimensional vector space to each vertex and a linear map to each arrow.

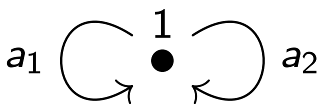
A *dimension vector* of Q is an element of the form $\beta \in \mathbb{Z}_{\geq 0}^{Q_0}$.

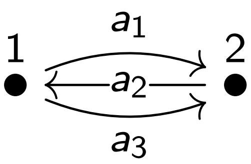
Given Q and β, the *representation space* is

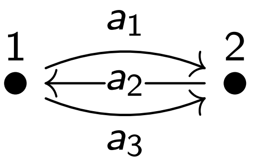
$$Rep(Q, \beta) := \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\beta_{t(a)}}, \mathbb{C}^{\beta_{h(a)}}).$$

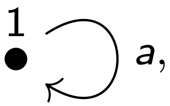
$Rep(Q, \beta)$ has a natural $\mathbb{G}_\beta$-action, where $\mathbb{G}_\beta := \prod_{i \in Q_0} GL_{\beta_i}$.

Background. Examples:

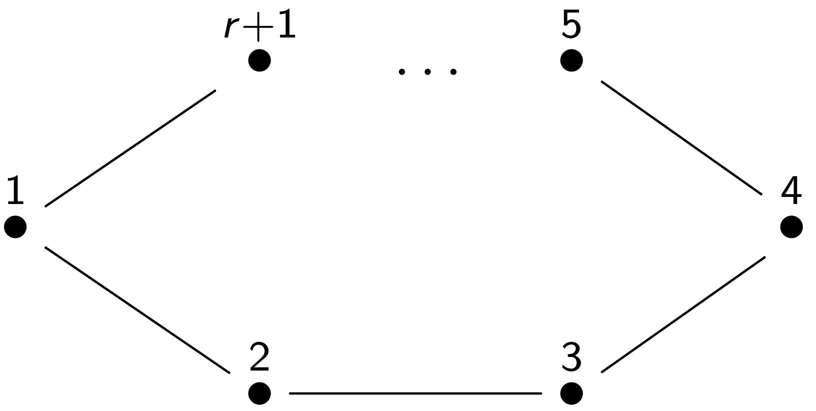
- 2-Jordan 

, 3-Kronecker 

,
- Cycle 

, loop 

- ADE-Dynkin if the underlying graph is of ADE-Dynkin type,
- Affine (type) $\widetilde{A}_r$-Dynkin



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Relations of quiver varieties to classical linear algebra: the study of GL_n-orbits on $\mathfrak{gl}_n = \mathrm{Lie}(GL_n)$.

Filtered quiver representations. Definitions:

Assume Q and β as before. Let

$$F_i^\bullet : \{0\} = U_i^0 \subseteq U_i^1 \subseteq U_i^2 \subseteq \ldots \subseteq U_i^N = \mathbb{C}^{\beta_i}$$

be a sequence of vector spaces, one for each $i \in Q_0$.
Then $F^\bullet Rep(Q, \beta)$ is a subspace of $Rep(Q, \beta)$ whose linear maps preserve the filtration of vector spaces at every level.
$F^\bullet Rep(Q, \beta)$ is called a *filtered quiver variety*.

Let $P_i \subseteq GL_{\beta_i}$ be the largest subgroup preserving the filtration of vector spaces at vertex i. Then $\mathbb{P}_\beta := \prod_{i \in Q_0} P_i$ acts on $F^\bullet Rep(Q, \beta)$ as a change-of-basis.

Filtered quiver representations. Examples:

- Q: $\begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd}$, $\beta = n$, $F^\bullet$ is the complete standard filtration of vector spaces. Then $F^\bullet Rep(Q, \beta) = \mathfrak{b}_n$, the space of $n \times n$ upper triangular matrices, with B-conjugation action on $\mathfrak{b}_n$, where $\mathrm{Lie}(B) = \mathfrak{b}_n$.
- Q: $\bullet \longrightarrow \bullet$, $\beta = (n, n)$, $F^\bullet$ is the complete standard filtration of vector spaces at each vertex. Then $F^\bullet Rep(Q, \beta) = \mathfrak{b}_n$ and $B_n \times B_n \subseteq GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acts on the filtered representation space via the left-right action.

Relations to classical (19^{th} century) linear algebra: the study of B-orbits on $\mathfrak{b} = \mathrm{Lie}(B)$.

Grothendieck-Springer resolution.

There are embeddings $B \overset{\psi}{\hookrightarrow} GL_n$ and $\mathfrak{b} \overset{\varphi}{\hookrightarrow} \mathfrak{gl}_n$ such that φ is ψ-equivariant. Consider

$$\begin{tikzcd}
 & & GL_n \times_B \mathfrak{b}. \arrow[dl, "\pi"'] \\
\mathfrak{b} \arrow[r] \arrow[loop below] & \mathfrak{gl}_n \arrow[loop below] & \\
B \arrow[r] & GL_n &
\end{tikzcd}$$

Lemma

Given $\widetilde{\mathfrak{gl}_n} := \{(x, \mathfrak{b}) \in \mathfrak{gl}_n \times GL_n/B : x \in \mathfrak{b}\}$, $\widetilde{\mathfrak{gl}_n} \cong GL_n \times_B \mathfrak{b}$.

Proof.

There is a map $(GL_n \times \mathfrak{b})/B \to \widetilde{\mathfrak{gl}_n}$, where $(g, x) \mapsto (gxg^{-1}, (g.B)/B)$. This map is GL_n-equivariant and is an isomorphism. $\square$

Grothendieck-Springer resolution.

Since GL_n acts on $\widetilde{\mathfrak{gl}_n}$ via $g.(x,\mathfrak{b}) = (xg^{-1}, g\mathfrak{b}g^{-1})$ and on $\mathfrak{gl}_n$ via the adjoint action, $\widetilde{\mathfrak{gl}_n} \to \mathfrak{gl}_n$ is a GL_n-equivariant map.

Lemma

There is a bijection between GL_n-orbits on $\widetilde{\mathfrak{gl}_n}$ and B-orbits on $\mathfrak{b}$.

Proof.

Consider $(GL_n \times \mathfrak{b})/B \to \mathfrak{b}/B$, where $(g,x) \mapsto gxg^{-1}$. This map is well-defined up to the B-conjugation action. Since the map is GL_n-equivariant, it descends to an isomorphism $\widetilde{\mathfrak{gl}_n}/GL_n \cong \mathfrak{b}/B$ as orbit spaces. □

Moral of the story: study the B-action on $\mathfrak{b}$.

Furthermore, $\mathbb{P}_\beta$-action on $F^\bullet Rep(Q,\beta)$ generalizes B-action on $\mathfrak{b}$ (21st century).

Comparing $\mathbb{U}_\beta$-invariants and $\mathbb{P}_\beta$-semi-invariants, where $\mathbb{U}_\beta$ is the maximal unipotent subgroup of $\mathbb{P}_\beta$.

Definition: $f \in \mathbb{C}[F^\bullet Rep(Q,\beta)]^{\mathbb{P}_\beta}$ is an *invariant polynomial* if $f(g.x) = f(x)$ for all $g \in \mathbb{P}_\beta$ and $x \in F^\bullet Rep(Q,\beta)$.

Definition: $f \in \mathbb{C}[F^\bullet Rep(Q,\beta)]^{\mathbb{P}_\beta,\chi}$ is a *semi-invariant polynomial* if $f(g.x) = \chi(g)f(x)$ for all $g \in \mathbb{P}_\beta$ and $x \in F^\bullet Rep(Q,\beta)$, where $\chi : \mathbb{P}_\beta \to \mathbb{C}^*$ is an algebraic group homomorphism, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The ring of semi-invariant polynomials is $\bigoplus_\chi \mathbb{C}[F^\bullet Rep(Q,\beta)]^{\mathbb{P}_\beta,\chi}$.

Semi-invariants under the $\mathbb{P}_\beta$-action are invariant for $\mathbb{U}_\beta$-action and $\mathbb{U}_\beta$-invariant polynomials that are homogeneous with respect to a generalized Q_0-grading are also semi-invariant (for some χ) for the $\mathbb{P}_\beta$-action. Thus, $\bigoplus_\chi \mathbb{C}[F^\bullet Rep(Q,\beta)]^{\mathbb{P}_\beta,\chi} \cong \mathbb{C}[F^\bullet Rep(Q,\beta)]^{\mathbb{U}_\beta}$.

$\mathbb{U}_\beta$-invariants and $\mathbb{P}_\beta$-semi-invariants: why are they interesting?

Use invariant and semi-invariant polynomials to construct new and interesting varieties; that is,

- construct the *affine quotient*

$$F^\bullet Rep(Q,\beta)/\!\!/\mathbb{P}_\beta := \operatorname{Spec}(\mathbb{C}[F^\bullet Rep(Q,\beta)]^{\mathbb{P}_\beta})$$

  of the vector space $F^\bullet Rep(Q,\beta)$ by $\mathbb{P}_\beta$,

- construct the *geometric (GIT) quotient*

$$F^\bullet Rep(Q,\beta)/\!\!/_{\chi}\mathbb{P}_\beta := \operatorname{Proj}(\bigoplus_{i\geq 0}\mathbb{C}[F^\bullet Rep(Q,\beta)]^{\mathbb{P}_\beta,\chi^i})$$

  of the space $F^\bullet Rep(Q,\beta)$ by $\mathbb{P}_\beta$ twisted by χ.

Filtered quiver varieties of finite *ADE*-Dynkin type.

Basic assumption: let $F^\bullet$ be the complete standard filtration of vector spaces at each vertex. Let $\mathfrak{t}_n \subseteq \mathfrak{gl}_n$ be the set of complex diagonal matrices.

Theorem (Im)

If Q is an ADE-Dynkin quiver and $\beta = (n, \ldots, n) \in \mathbb{Z}_{\geq 0}^{Q_0}$, then $\mathbb{C}[F^\bullet Rep(Q, \beta)]^{\mathbb{U}_\beta} \cong \mathbb{C}[\mathfrak{t}_n^{\oplus Q_1}]$.

If Q is an ADE-Dynkin quiver, then all classical semi-invariant techniques are applicable!

Classical techniques (for reductive groups) given by

- Schofield-van den Bergh (1999)
- Derksen-Weyman (2000)
- Domokos-Zubkov (2001)

are applicable.

Pathways. Definitions:

A *nontrivial path* is a sequence $a_m \cdots a_1$ of arrows such that $t(a_{i+1}) = h(a_i)$ for all $1 \leq i < m$. We write e_i as the *trivial (empty) path* at vertex i.

The *path algebra* $\mathbb{C}Q$ of Q is the $\mathbb{C}$-algebra with basis the paths in Q, with the product of two paths p and q given by $p \circ q = pq$ if $t(p) = h(q)$; otherwise, $p \circ q = 0$.

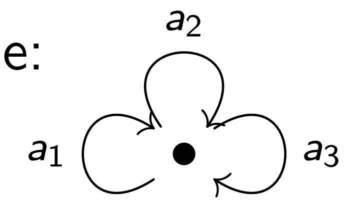
A *relation* of a quiver Q is a subspace of $\mathbb{C}Q$ spanned by linear combinations of paths having a common source and a common target, and of length at least 2 (Michel Brion).

A *quiver with relations* is a pair (Q, I), where Q is a quiver and I is a two-sided ideal of $\mathbb{C}Q$ generated by relations.

The *quotient algebra* $\mathbb{C}Q/I$ is the path algebra of (Q, I).

Pathways.

Example:

Let Q: 

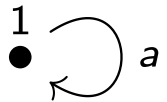
. Then $\mathbb{C}Q = \mathbb{C}\langle a_1, a_2, a_3 \rangle$.
Let I be the ideal generated by $a_i a_j - a_j a_i$, $1 \leq i < j \leq 3$.
Then $\mathbb{C}Q/I = \mathbb{C}[a_1, a_2, a_3]$.

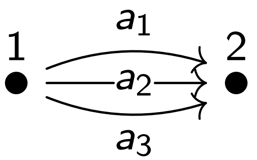
Definition: a path p is *reduced* if it is the class $[p] \neq 0$ in $\mathbb{C}Q/\langle q^2 : q \in \mathbb{C}Q, l(q) \geq 1 \rangle$, where $l(q)$ is the number of arrows in q.

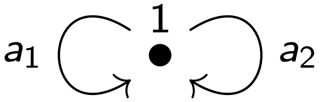
Definition: a *pathway* from vertex i to vertex j is a reduced path from i to j. We define *pathways* of a quiver Q to be the set of all pathways from vertex i to vertex j, where $i, j \in Q_0$.

More examples on the next slide.

Pathways. Examples:

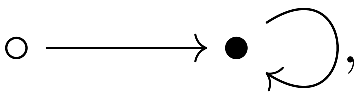
- Q: 

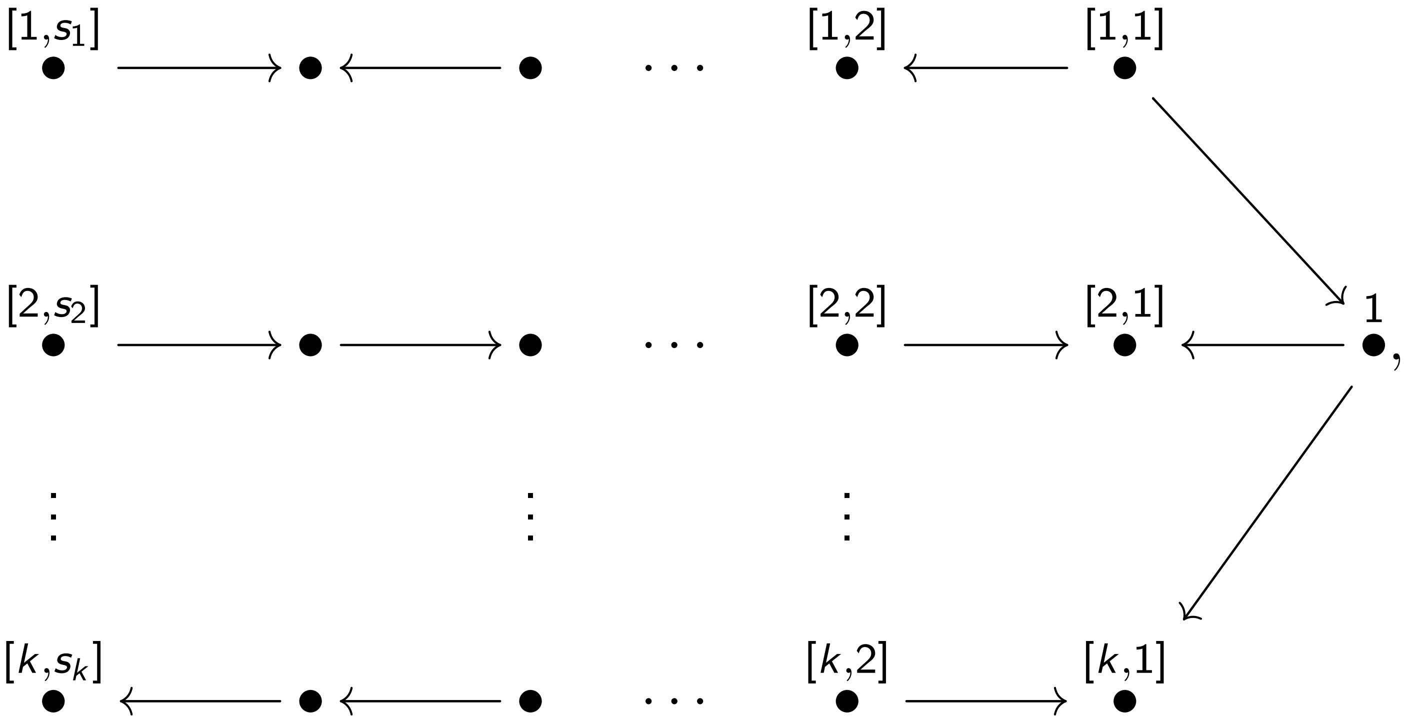
.
Paths of Q consist of $e_1, a, a^2, a^3, \ldots$.
Pathways of Q consist of e_1 and a.
This quiver has at most 2 pathways.
- Q: 

.
There is one pathway from vertex 1 to vertex 1: e_1.
There are 3 pathways from vertex 1 to vertex 2: a_1, a_2, a_3.
There is one pathway from vertex 2 to vertex 2: e_2.
This quiver has at most 3 pathways between any two vertices.
- Q: 

.
Pathways consist of $e_1, a_1, a_2, a_1a_2, a_2a_1, a_1a_2a_1, a_2a_1a_2$.
This Q has more than 2 pathways.

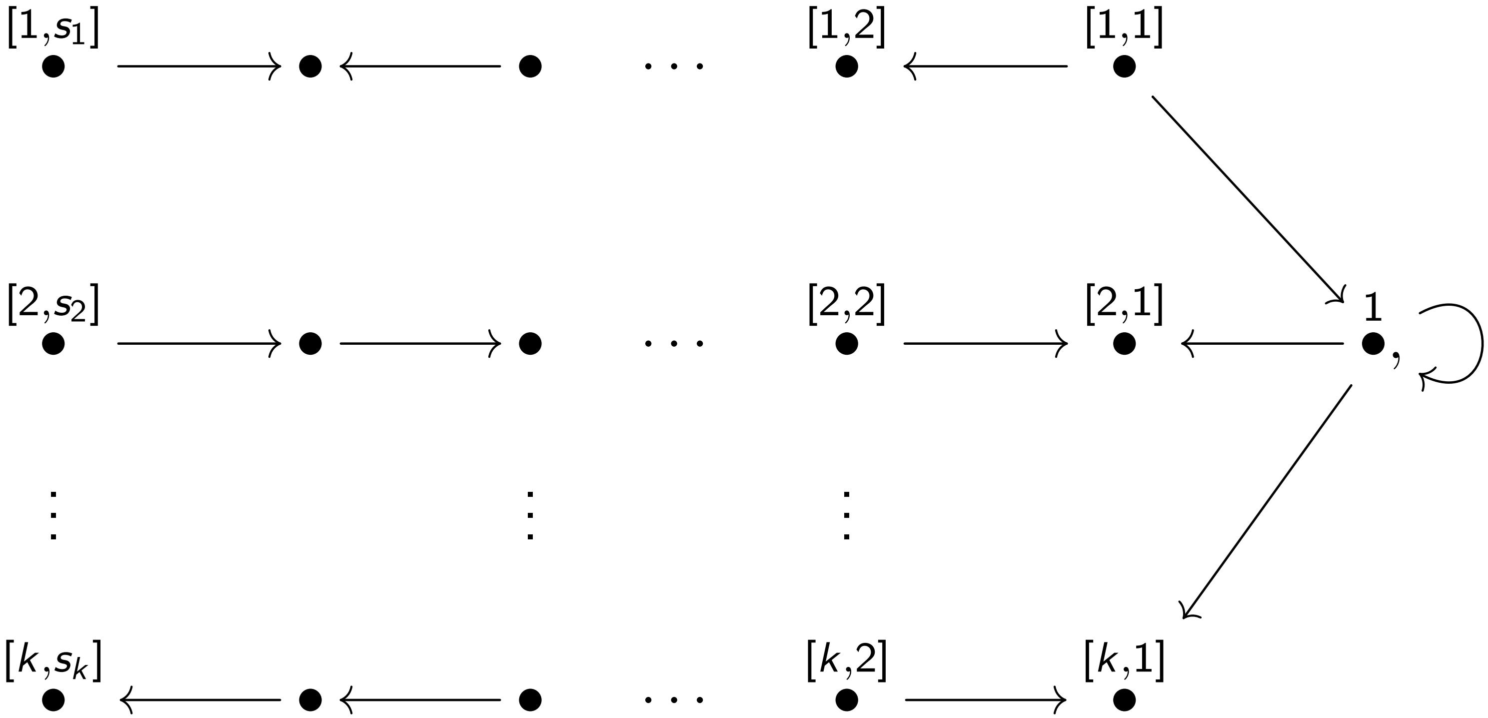
Examples of quivers with at most two pathways between any two vertices.

- ADE-Dynkin quivers,
- Framed $\widetilde{A}_r$-Dynkin quivers, which includes ,
- Star-shaped quivers have k legs (each of length s_k), e.g.,



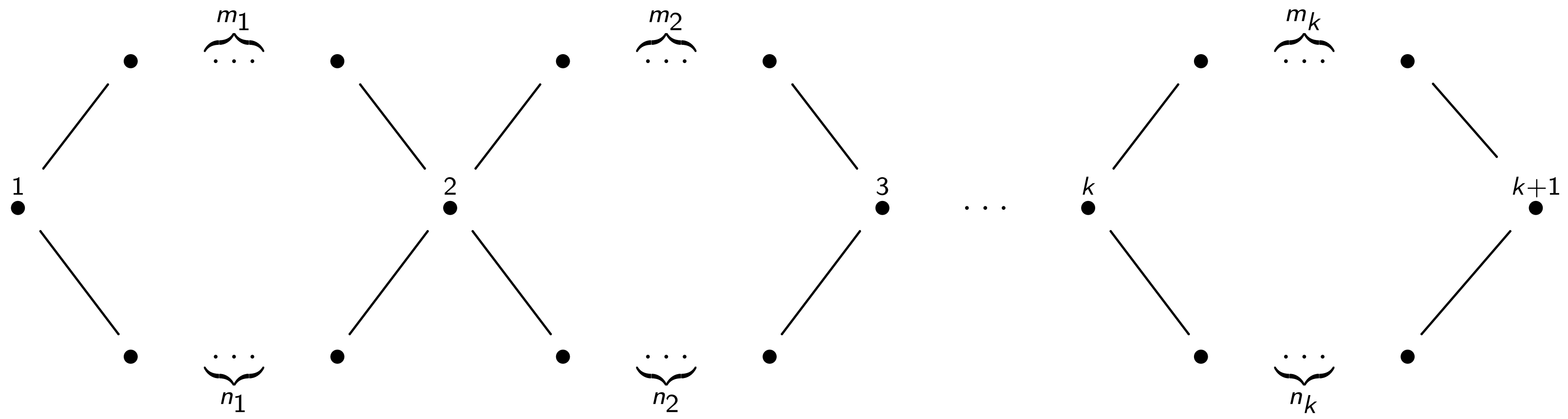
Examples of quivers with at most two pathways.

- Comet-shaped quivers have k legs (each of length s_k), with 1 loop on the central vertex, e.g.,



Examples of quivers with at most two pathways.

- A quiver whose underlying graph is



with the condition that vertices $2, 3, \dots, k$ are a source or a sink (in any order).

Quivers with at most two pathways.

Basic assumption: let $F^\bullet$ be the complete standard filtration of vector spaces at each nonframed vertex. Let $\mathfrak{t}_n \subseteq \mathfrak{gl}_n$ be the set of complex diagonal matrices, and let $\mathbb{U}_\beta := U^{Q_0} \subseteq B^{Q_0}$, $B \subseteq GL_n(\mathbb{C})$ is subgroup of invertible upper triangular matrices.

Theorem (Im)

Let Q be a quiver and let $\beta = (n, \ldots, n)$. Then Q is a nonframed quiver with at most two distinct pathways between any two vertices if and only if $\mathbb{C}[F^\bullet Rep(Q, \beta)]^{\mathbb{U}_\beta} \cong \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$, where Q_1 is the set of arrows whose tail and head are nonframed vertices.

Remark: want to use classical techniques to obtain semi-invariants for certain filtered quiver representations? Then restrict to quivers with at most 2 pathways between any two of its vertices.

Quivers with at most two pathways.

That is, suppose Q is a nonframed quiver. Then

Q has at most two distinct pathways between any two vertices
if and only if
$\mathbb{C}[F^\bullet Rep(Q,\beta)]^{\mathbb{U}_\beta} \cong \mathbb{C}[\mathfrak{t}^{\oplus Q_1}]$, where Q_1 is the number of arrows of Q.

Sketch of proof: similar to the proof for filtered quiver varieties for finite ADE-Dynkin type.

Hamiltonian reduction and reduction by stages

We will now discuss Hamiltonian reduction and reduction by stages.

Construction of $T^*(\mathfrak{b} \times \mathbb{C}^n)$, cf. Nevins' manuscript.

Consider B-action on $\mathfrak{b} \times \mathbb{C}^n$ via $b.(r, i) = (brb^{-1}, bi)$ and $G \times B$-action on $G \times \mathfrak{b} \times \mathbb{C}^n$ via

$$g.(g', r, i) = (g'g^{-1}, r, gi) \text{ and } b.(g', r, i) = (g'b^{-1}, brb^{-1}, i),$$
$$\text{where } b \in B \text{ and } g \in G.$$

This gives two moment maps

$$T^*(\mathfrak{b} \times \mathbb{C}^n) \xrightarrow{\mu_B} \mathfrak{b}^* \cong \mathfrak{gl}_n/\mathfrak{u}^+, \ (r, s, i, j) \mapsto [r, s] + ij, \text{ and}$$
$$T^*(G \times \mathfrak{b} \times \mathbb{C}^n) \xrightarrow{\mu_{G\times B}} \mathfrak{g}^* \times \mathfrak{b}^*, \ (g', \theta, r, s, i, j) \mapsto (\theta - ij, [r, s] + \overline{\theta}),$$
$$\text{where } \overline{\theta} : \mathfrak{gl}_n \to \mathfrak{b}^*.$$

There is a bijection of B-orbits on $\mu_B^{-1}(0)$ and $G \times B$-orbits on $\mu_{G\times B}^{-1}(0)$:

$$\mu_B^{-1}(0)/B \cong \mu_{G\times B}^{-1}(0)/G \times B \cong T^*(G \times \mathfrak{b} \times \mathbb{C}^n/G \times B)$$
$$\cong T^*(G \times_B \mathfrak{b} \times \mathbb{C}^n/G) = T^*(\widetilde{\mathfrak{g}} \times \mathbb{C}^n/G).$$

Results for $T^*(\mathfrak{b} \times \mathbb{C}^n)$.

We will thus study $\mu_B^{-1}(0)/B$.

Definition: an element of $\mathfrak{b}$ is *regular* if its stabilizer dimension is minimal. An element in $\mathfrak{b}$ is *semisimple* if it is diagonalizable.

Let $\mu_B^{-1}(0)^{rss}$ be the restriction of $\mu_B^{-1}(0)$ to the regular semisimple locus, i.e., it is the locus where eigenvalues of $\mathfrak{b}$ are pairwise distinct.

Proposition (Im)

$$\mu_B^{-1}(0)^{rss} /\!\!/ B \cong \mathbb{C}^{2n} \setminus \Delta_n, \text{ where}$$
$$\Delta_n = \{(x_1, \ldots, x_n, 0, \ldots, 0) : x_i = x_j \, \forall \, i \neq j\}.$$

Open problems for the Hamiltonian reduction of the cotangent bundle of the enhanced Grothendieck-Springer resolution.

Nevins' Conjecture: $\mu_B^{-1}(0)$ is a complete intersection.

Study

$$\begin{tikzcd}
 & \mu_B^{-1}(0) \arrow[dl] \arrow[dr] & \\
\mu_B^{-1}(0)/\!\!/_{\chi} B \arrow[rr, dashed, "\psi_{\chi,\chi'}"] \arrow[dr] & & \mu_B^{-1}(0)/\!\!/_{\chi'} B. \arrow[dl] \\
 & \mu_B^{-1}(0)/\!\!/ B &
\end{tikzcd}$$

Why? What is the motivation?

Motivation for open problems.

Let $G := GL_n(\mathbb{C})$ act on $M_n \times \mathbb{C}^n$.

Theorem (Crawley-Boevey, Gan-Ginzburg)

$\mu^{-1}(0)$ *is a complete intersection with* $n+1$ *irreducible components.*

Theorem (Nakajima)

$$\begin{tikzcd}
(\mathbb{C}^2)^{[n]} \cong \mu^{-1}(0)/\!\!/_{\det} G \arrow[rr, dashed, "\cong"] \arrow[dr] & & \mu^{-1}(0)/\!\!/_{\det^{-1}} G \arrow[dl] \\
& \mu^{-1}(0)/\!\!/ G \cong S^n(\mathbb{C}^2) &
\end{tikzcd}$$

Further generalizations.

Consider the following substitutions:

$$\mathfrak{gl}_n \rightsquigarrow Rep(Q,\beta), \quad \mathfrak{b} \rightsquigarrow F^\bullet Rep(Q,\beta)$$
$$GL_n \rightsquigarrow \mathbb{G}_\beta := \prod_{i\in Q_0} GL_{\beta_i}, \quad B \rightsquigarrow \mathbb{P}_\beta := \prod_{i\in Q_0} P_i.$$

Thus replace $\widetilde{\mathfrak{gl}_n} \cong GL_n \times_B \mathfrak{b}$ with $\widetilde{Rep(Q,\beta)} := \mathbb{G}_\beta \times_{\mathbb{P}_\beta} F^\bullet Rep(Q,\beta)$, and since we have natural projection maps

$$\begin{tikzcd} & \widetilde{Rep(Q,\beta)} \arrow[dl] \arrow[dr] & \\ \mathbb{G}_\beta/\mathbb{P}_\beta & & Rep(Q,\beta), \end{tikzcd}$$

study the moment map $T^*F^\bullet Rep(Q,\beta) \xrightarrow{\mu_{\mathbb{P}_\beta}} \mathrm{Lie}(\mathbb{P}_\beta)^*$ for $\mathbb{P}_\beta$-action on $F^\bullet Rep(Q,\beta)$.

The end.

Thank you.