EQUIVALENCE OF DOMAINS WITH ISOMORPHIC SEMIGROUPS OF ENDOMORPHISMS

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Abstract. For two bounded domains $\Omega_1$, $\Omega_2$ in $\mathbb{C}$ whose semigroups of analytic endomorphisms $E(\Omega_1)$, $E(\Omega_2)$ are isomorphic with an isomorphism $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$. Eremenko proved in 1993 that there exists a conformal or anticonformal map $\psi : \Omega_1 \rightarrow \Omega_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(\Omega_1)$.
In the present paper we prove an analogue of this result for the case of bounded domains in $\mathbb{C}^n$.

1. Introduction

A classical theorem of L. Bers says that every $\mathbb{C}$-algebra isomorphism $H(A) \rightarrow H(B)$ of algebras of holomorphic functions in domains $A$ and $B$ in the complex plane has the form $f \mapsto f \circ \theta$, where $\theta : B \rightarrow A$ is a conformal isomorphism, or $f \mapsto \overline{f} \circ \theta$ with anticonformal $\theta$. In particular, the algebras $H(A)$ and $H(B)$ are isomorphic if and only if the domains $A$ and $B$ are conformally equivalent. H. Iss’sa [9] obtained a similar theorem for fields of meromorphic functions on Stein spaces. A good reference for these results is [5].

Likewise, a question of recovering a topological space from the algebraic structure of its semigroup of continuous self-maps has been extensively studied [12].

In 1990, L. Rubel asked whether similar results hold for semigroups (under composition) $E(D)$ of holomorphic endomorphisms of a domain $D$. A. Hinkkanen constructed examples [6] which show that even non-homeomorphic domains in $\mathbb{C}$ can have isomorphic semigroups of endomorphisms. The reason is that the semigroup of endomorphisms of a domain can be too small to characterize this domain.

However, in 1993, A. Eremenko [4] proved that for two Riemann surfaces $D_1$, $D_2$, which admit bounded nonconstant holomorphic functions, and such that the semigroups of analytic endomorphisms $E(D_1)$ and $E(D_2)$ are isomorphic, and an isomorphism $\varphi : E(D_1) \rightarrow E(D_2)$, there exists a conformal or anticonformal map $\psi : D_1 \rightarrow D_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(D_1)$. In the present paper we investigate the analogue of this result for the case of bounded domains in $\mathbb{C}^n$. The theorems of Bers and Iss’sa, mentioned above, do not extend to arbitrary domains in $\mathbb{C}^n$.

For a bounded domain $\Omega$ in $\mathbb{C}^n$ we denote by $E(\Omega)$ the semigroup of analytic endomorphisms of $\Omega$ under composition. In what follows, we say that a map is (anti-) biholomorphic, if it is biholomorphic or anti-biholomorphic. We prove the following theorem.

Theorem 1. Let $\Omega_1$, $\Omega_2$ be bounded domains in $\mathbb{C}^n$, $\mathbb{C}^m$ respectively, and suppose that there exists $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$, an isomorphism of semigroups. Then $n = m$.

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and there exists an (anti-) biholomorphic map $\psi : \Omega_1 \to \Omega_2$ such that

$$\varphi f = \psi \circ f \circ \psi^{-1}, \text{ for all } f \in E(\Omega_1).$$

The existence of a homeomorphism $\psi$ satisfying (1) follows from simple general considerations (Section 2). The hard part is proving that $\psi$ is (anti-) biholomorphic. In dimension 1 this is done by linearization of holomorphic germs of $f \in E(\Omega)$ near an attracting fixed point. In several dimensions such linearization theory exists ([1], pp. 192–194), but it is too complicated (many germs with an attracting fixed point are non-linearizable, even formally). In Sections 3, 4 we show how to localize the problem. In Sections 5, 6 we describe, using only the semigroup structure, a large enough class of linearizable germs. Linearization of these germs permits us to reduce the problem to a matrix functional equation, which is solved in Section 7.

In Section 8 we complete the proof that $\psi$ is (anti-) biholomorphic.

Theorem 1 can be slightly generalized, namely one may assume that $\varphi$ is an epimorphism. In Section 9 we prove the following theorem.

**Theorem 2.** If $\varphi : E(\Omega_1) \to E(\Omega_2)$ is an epimorphism between semigroups, where $\Omega_1$, $\Omega_2$ are bounded domains in $\mathbb{C}^n$, $\mathbb{C}^n$ respectively, then $\varphi$ is an isomorphism.

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2. **Topography**

For a bounded domain $\Omega$ in $\mathbb{C}^n$ we denote by $C(\Omega)$ the subsemigroup of $E(\Omega)$ consisting of constant maps. An endomorphism $c_z$ is constant if it sends $\Omega$ to a point $z \in \Omega$. The subset $C(\Omega) \subset E(\Omega)$ can be described using only the semigroup structure as follows:

$$c \in C(\Omega) \iff \forall (f \in E(\Omega)), \quad (c \circ f = c).$$

It is clear that we have a bijection between constant endomorphisms of $\Omega$ and points of this domain as a set: to each $z$ corresponds a unique $c_z \in C(\Omega)$ and vice versa, so we can identify the two. Under this identification, a subset of $\Omega$ corresponds to a subsemigroup of $C(\Omega)$.

Having defined points of a domain in terms of its semigroup structure of analytic endomorphisms, we can construct a map $\psi$ between $\Omega_1$ and $\Omega_2$ as follows

$$\psi(z) = w \iff \varphi c_z = c_w.$$ 

So defined, $\psi$ satisfies (1). Indeed, let $f \in E(\Omega_1)$, $f(z) = \zeta$. This is equivalent to

$$f \circ c_z = c_{\zeta}.$$ 

Applying $\varphi$ to both sides of (4) we have

$$\varphi f \circ c_{\psi(z)} = c_{\varphi(\zeta)}.$$ 

But (5) is equivalent to $\varphi f(\psi(z)) = \psi(\zeta) = \psi(f(z))$, which is (1).
We describe the topology of a domain $\Omega$ using its injective endomorphisms. A map $f \in E(\Omega)$ is injective if and only if
\[\forall(c' \in C(\Omega)) \forall(c'^\prime \in C(\Omega)), \quad ((f \circ c' = f \circ c'^\prime) \Rightarrow (c' = c'^\prime)).\]

We denote the class of injective endomorphisms of $\Omega$ by $E_i(\Omega)$. For every $f \in E_i(\Omega)$, $f_\Omega$ is open [2]. The family $\{f_\Omega \in E_i(\Omega)\}$ of subsets of $\Omega$ forms a base of topology, because every $z \in \Omega$ has a neighborhood $f(\Omega)$, where $f(\zeta) = z + \lambda(\zeta - z)$, $f$ belongs to $E_i(\Omega)$ for every $\lambda$ such that $|\lambda|$ is small.

To summarize, we described subsets of $\Omega$ and the topology on it using only the semigroup structure of $E(\Omega)$. Since this is so, the semigroup structure also defines the notions of an open set, closed set, compact set, closure of a set.

Now we can easily prove continuity of the map $\psi$ constructed above. Indeed, let $g(\Omega_2)$, $g \in E_i(\Omega_2)$ be a set from the base of topology of $\Omega_2$. We take $f = \varphi^{-1} g$. Then $f \in E_i(\Omega_1)$ and $\psi^{-1}(g(\Omega_2)) = f(\Omega_1)$, which proves that $\psi$ is continuous. Since $\varphi$ is an isomorphism, the same argument works to prove that $\psi^{-1}$ is also continuous, and thus $\psi$ is a homeomorphism.

Therefore the domains $\Omega_1$, $\Omega_2$ are homeomorphic, and hence [8] they have the same dimension, i. e. $n = m$.

3. LOCALIZATION

We need the following lemma.

**Lemma 1.** Suppose $H$ is a semigroup with identity, and $f$ an element of $H$ with the following two properties:
(i) $hf = fh$, for every $h$ in $H$;
(ii) $h_1 f = h_2 f$ implies $h_1 = h_2$, for every $h_1$ and $h_2$ in $H$.

Then there exists a semigroup $S_f$ and a monomorphism $i : H \to S_f$, such that $i(f)$ is invertible in $S_f$ and commutes with all elements of $S_f$. Moreover, the semigroup $S_f$ satisfies the following universal property: for every semigroup $S_1$ with a monomorphism $i_1 : H \to S_1$ such that $i_1(f)$ is invertible in $S_1$ and commutes with all elements of $S_1$, there exists a unique monomorphism $i : S_f \to S_1$ such that $i_1 = i \circ i$.

**Remark 1.** Uniqueness of $i_1$ implies that the semigroup $S_f$ with the universal property is unique up to an isomorphism.

**Proof.** We construct $S_f$ as follows. First we consider formal expressions of the form $h f^k$, where $h \in H$ and $k$ is an integer (may be positive, negative or zero). Then we define a multiplication on this set: $h_1 f^{k_1} \cdot h_2 f^{k_2} = h_1 h_2 f^{k_1+k_2}$. Next we consider a relation on the set of formal expressions: $h_1 f^{k_1} \sim h_2 f^{k_2}$ if $k_1 \leq k_2$ and $h_1 = h_2 f^{k_2-k_1}$ in $H$, or $k_2 \leq k_1$ and $h_2 = h_1 f^{k_1-k_2}$ in $H$. It is easy to verify that this is an equivalence relation and it is compatible with the operation $\cdot$; that is, $x \sim y \Rightarrow v \sim u \Rightarrow u \sim v$.

Lastly, let $S_f$ be the set of equivalence classes with the binary operation induced by $\cdot$. For $S_f$ to be a semigroup, we need to show that the binary operation $\cdot$ is associative. Let $h_1 f^{k_1} \sim h_1' f^{k_1'}$, $h_2 f^{k_2} \sim h_2' f^{k_2'}$ and $h_3 f^{k_3} \sim h_3' f^{k_3'}$. We need to show that $(h_1 f^{k_1} \cdot h_2 f^{k_2}) \sim (h_1' f^{k_1'} \cdot h_2' f^{k_2'})$. By the definition of the operation $\cdot$, the last equivalence is the same as $h_1 h_2 h_3 f^{k_1+k_2+k_3} \sim h_1' h_2' h_3' f^{k_1'+k_2'+k_3'}$. Assuming that $k_1 + k_2 + k_3 \leq k_1' + k_2' + k_3'$, we have essentially one possibility to consider (the others are either similar or trivial): $k_1 \leq k_1'$, $k_2 \leq k_2'$, $k_3 \leq k_3'$. In
this case $h_1 h_2 h_3 f^{k_3} = h_1' h_2' h_3' f^{k_3'}$. Now we can use the cancellation property (ii) to get the desired equivalence.

The semigroup $H$ is embedded into $S_f$ via $i : h \mapsto [h f^0]$. The element $i(f) = [id f]$, where $id$ is the identity in $H$, is invertible in $S_f$ with the inverse $[id f^{-1}]$. Clearly, $[id f]$ commutes with all elements of $S_f$.

Now, suppose that $S_1, i_1 : H \to S_1$ is a semigroup and a monomorphism, such that $i_1(f)$ is invertible in $S_1$ and commutes with all elements of $S_1$. Then we define

$$i_1([h f^k]) = i_1(h)(i_1(f))^k.$$ 

This definition does not depend on a representative of $[h f^k]$. Indeed, suppose $h_1 f^{k_1} \sim h_2 f^{k_2}$ and assume $k_1 \leq k_2$. Then $h_1 = h_2 f^{k_2-k_1}$, and thus $i_1(h_1) = i_1(h_2) i_1(f)^{k_2-k_1}$. Hence $i_1(h_1) i_1(f)^{k_1} = i_1(h_2) i_1(f)^{k_2}$.

So defined, $i_1$ is a homomorphism:

$$\hat{i}_1([h f^k]) = i_1(h_1 h_2 f^{k_1+k_2})$$
$$= i_1(h_1) i_1(h_2) i_1(f)^{k_1+k_2} = i_1(h_1) i_1(h_2) i_1(f)^{k_1} i_1(f)^{k_2}$$
$$= i_1(h_1) i_1(f)^{k_1} i_1(h_2) i_1(f)^{k_2}.$$

The relation $\hat{i}_1 \circ i = i_1$ holds, since $\hat{i}_1([h f^0]) = i_1(h)$ for all $h \in H$.

Uniqueness of $\hat{i}_1$ is clear. Lemma 1 is proved.

4. Extension of $\varphi$

Following [4], we say that for a bounded domain $\Omega$ an element $f \in E(\Omega)$ is good at $z \in \Omega$, denoted by $f \in G_z(\Omega)$, if

1. $z$ is a unique fixed point of $f$;
2. $f(\Omega)$ has compact closure in $\Omega$;
3. $f$ is injective in $\Omega$.

Property 3 of a good element was already stated in terms of the semigroup structure of $\Omega$. Since the topology on $\Omega$ was described using only the semigroup structure, Property 2 can also be stated in these terms. Property 1 can be expressed in terms of the semigroup structure as

$$(f \circ c_z = c_z) \land ((f \circ c_\zeta = c_\zeta) \Rightarrow (c_\zeta = c_z)).$$

Since $f$ is an endomorphism of a domain, all eigenvalues $\lambda$ of its linear part at $z$ satisfy $|\lambda| \leq 1$ [10]. Moreover, $|\lambda| < 1$ because the closure of $f(\Omega)$ is a compact set in $\Omega$. The injectivity of $f$ implies [2] that it is biholomorphic onto $f(\Omega)$ and the Jacobian determinant of $f$ does not vanish at any point of $\Omega$.

It is clear that for every $z \in \Omega$ a good element $f$ at $z$ exists. For example, we can take $f(\zeta) = z + \lambda(\zeta - z)$ with sufficiently small $|\lambda|$.

Consider a good element $f \in G_z(\Omega)$ and its commutant $H_f(\Omega)$ in $E(\Omega)$:

$$H_f(\Omega) = \{ h \in E(\Omega) : h f = f h \}.$$ 

Clearly $H_f(\Omega)$ is a subsemigroup of $E(\Omega)$. The element $f$, being good (hence injective), satisfies the cancellation property (ii) of Lemma 1 in $H_f(\Omega)$. Thus, by Lemma 1, we have the extension $S_f$ of $H_f(\Omega)$ in which $f$ is invertible and commutes with all elements of $S_f$. In the case of analytic endomorphisms we can embed $H_f(\Omega)$ into the subsemigroup of $A_z$, the semigroup of germs of analytic mappings at $z$ under composition, consisting of elements that commute with the germ of $f$. 

and containing the germ of $f^{-1}$. We use the universal property of Lemma 1 to conclude that $S_f$ is isomorphic to a subsemigroup of $A_z$. We identify $S_f$ with this semigroup, i.e., we consider elements of $S_f$ as germs of analytic mappings at $z$.

In proving that $\psi$ is (anti-) biregular we need to show that it is so in a neighborhood of every point of $\Omega_1$. Since an (anti-) biregular type of a domain is preserved by translations in $\mathbb{C}^n$, it is enough to show that $\psi$ is (anti-) biregular in a neighborhood of $0 \in \mathbb{C}^n$, assuming that $\Omega_1$ and $\Omega_2$ contain $0$ and $\psi(0) = 0$.

Let $\varphi : E(\Omega_1) \to E(\Omega_2)$ be an isomorphism of the semigroups, $f$ a good element, $f \in G_0(\Omega_1)$, and $H_f(\Omega_1)$ the commutant of $f$. Then clearly $H_\varphi(\Omega_2) = \varphi(H_f(\Omega_1))$ is the commutant of $g = \varphi f$. By Lemma 1, we have the extensions $S_f$, $S_g$ of $H_f(\Omega_1)$ and $H_\varphi(\Omega_2)$ respectively, and by the universal property of this lemma the isomorphism $\varphi$ extends to an isomorphism

$$\Phi : S_f \to S_g.$$ 

5. System of projections and linearization

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. We say that a good element $f \in G_0(\Omega)$ is very good at $0$, and write $f \in VG_0(\Omega)$, if the corresponding semigroup $S_f \subset A_0$ constructed in Section 4 contains a system of elements, which we call a system of projections, $\{p_i\}_{i=1}^n$ with the following properties:

(a) $\forall (i = 1, \ldots, n), \quad (p_i \neq 0)$;
(b) $\forall (i = 1, \ldots, n), \quad (p_i^2 = p_i)$;
(c) $\forall (i, j = 1, \ldots, n, i \neq j), \quad (p_ip_j = 0)$.

There does exist a very good element, since we can take $f$ to be a homothetic transformation at $0$ with sufficiently small coefficient, $P_i$ a projection on the $i$'th coordinate of the standard coordinate system. Clearly, $p_if = p_i$ and there exists $k$ such that $p_i^kf = E(\Omega)$, and hence $p_i \in S_f$. From now on, we fix a very good element $f \in VG_0(\Omega)$, associated semigroup $H_f(\Omega)$, $S_f$ and a system of projections $\{p_i\}$.

We introduce another subsemigroup of $E(\Omega)$:

$$P_f(\Omega) = \{h \in G_0(\Omega) \cap H_f(\Omega), \quad hp_i = p_ih \quad i = 1, \ldots, n\},$$

where the commutativity relations are in $S_f \subset A_0$. Notice that $P_f(\Omega) \neq \emptyset$ since $f$ belongs to it.

Lemma 2. For every $h \in P_f(\Omega)$ there exists a biregular germ $\theta_h$ at $0 \in \mathbb{C}^n$ such that $\theta_h h = \Lambda h$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_\cdot)$ is an invertible diagonal matrix which is similar to $dh(0)$ in $GL(n, \mathbb{C})$.

Proof. The relations $p_i \neq 0$, $p_i^2 = p_i$, $p_ip_j = 0$, $i \neq j$, imply that for $P_i = dp_i(0)$, the linear part of $p_i$ at $0$, we have $P_i \neq 0$, $P_i^2 = P_i$, $P_ip_j = 0$, $i \neq j$. Since the matrices $P_i$ commute, there exists [7] a matrix $A \in GL(n, \mathbb{C})$ such that $P_i^2 = AP_iA^{-1} = \Delta_i = \text{diag}(0, 0, \ldots, 1, \ldots, 0)$, where the only non-zero entry appears in the $i$'th place.

Since $P_i^2 = p_i$, $i = 1, \ldots, n$, we can use the argument given in [10] to linearize $p_i$, i.e., there exists a biregular germ $\xi_i$ at $0$ such that $\xi_ip_i = P_i\xi_i$, $d\xi_i(0) = \text{id}$, $i = 1, \ldots, n$. The map $\xi_i$ is constructed in [10] as follows:

$$\xi_i = \text{id} + (2P_i - \text{id})(p_i - P_i), \quad i = 1, \ldots, n.$$
If we take $\xi^j_i = A\xi_i$, we have $\xi_i P_i = P_i^j \xi^j_i$. For simplicity of notations, we assume that $\xi_i$ itself conjugates $p_i$ to a diagonal matrix, that is, $P_i = P_i^j$ (in this case $P_i$ is not necessarily $dp_i(0)$, but rather $Adp_i(0)A^{-1}$; $d\xi_i(0) = A$). For every $i = 1, \ldots, n$ we have $h_i P_i = P_i h_i$, where $h_i = \xi_i h h_i^{-1}$. Let $H_i = dh_i(0)$. Then $H_i P_i = P_i H_i$, and hence in the $i$'th row and the $i$'th column the matrix $H_i$ has only one non-zero entry, $\lambda_i$, which is located at their intersection. Thus $\lambda_i$ has to be an eigenvalue of $H_i$, and hence of the linear part of $h$. In particular, $0 < |\lambda_i| < 1$.

Let $I_i : \mathbb{C} \to \mathbb{C}^n$ be the embedding $z \mapsto (0, \ldots, z, \ldots, 0)$, where the only non-zero entry is $z$, which is in the $i$'th place; and $\pi_i : \mathbb{C}^n \to \mathbb{C}$ a projection $(z_1, \ldots, z_n) \mapsto z_i$, corresponding to the $i$'th axis. For every $i = 1, \ldots, n$, the map $\pi_i h_i I_i$ sends a neighborhood of $0$ in $\mathbb{C}$ into $\mathbb{C}$ and its derivative at $0$, $\lambda_i$, is an eigenvalue of $h$. Hence ([3], p. 31) $\pi_i h I_i$ is linearized by the unique solution $\eta h, i$ of the Schröder equation

$$\eta(h_i I_i) = \lambda_i \eta_i, \quad \eta(0) = 0, \quad \eta'(0) = 1.$$

Since $P_i I_i = I_i$, $\pi_i P_i I_i = id_{\mathbb{C}}$, we can rewrite (6) as

$$\eta_h, i \pi_i h P_i I_i = \lambda_i \eta_h, i \pi_i P_i I_i, \text{ or } \eta_h, i \pi_i h P_i = \lambda_i \eta_h, i \pi_i P_i.$$

But $h_i P_i = P_i h_i$, and so

$$\eta_h, i \pi_i h P_i = \lambda_i \eta_h, i \pi_i P_i.$$

The equation (7), in its turn, is equivalent to

$$\eta_h, i \pi_i \xi_i P_i = \lambda_i \eta_h, i \pi_i \xi_i P_i.$$

We denote

$$\theta_{h, i} = \eta_h, i \pi_i \xi_i P_i,$$

a map from a neighborhood of $0 \in \mathbb{C}^n$ into $\mathbb{C}$. Then (8) becomes $\theta_{h, i} h = \lambda_i \theta_{h, i}$. Now we define

$$\theta_h = (\theta_{h, 1}, \ldots, \theta_{h, n}),$$

which is a germ of an analytic map at $0$. This germ linearizes $h$:

$$\theta_h h = (\theta_{h, 1} h, \ldots, \theta_{h, n} h) = (\lambda_1 \theta_{h, 1}, \ldots, \lambda_n \theta_{h, n}) = \Lambda \theta_h,$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is an invertible diagonal matrix, which has eigenvalues of $dh(0)$ on its diagonal.

The germ $\theta_h$ is biholomorphic. Indeed,

$$\theta_{h, i} = \eta_h, i \pi_i \xi_i P_i = \eta_h, i \pi_i P_i \xi_i, \quad i = 1, \ldots, n.$$

Using the chain rule, we see that $d\theta_h(0) = A$, where $A$ is an invertible diagonal matrix that diagonalizes $P_i$. We conclude that $\theta_h$ is biholomorphic. Lemma 2 is proved.

6. Simultaneous linearization

Using Lemma 2, we can linearize elements of $P_f(\Omega)$. Namely, for every $h \in P_f(\Omega)$ there exists $\theta_h$ (constructed in Section 5), such that $\theta_h h = \Lambda h \theta_h$, where $\Lambda$ is an invertible diagonal matrix. In particular, we can linearize $f$:

$$\theta_f f = \Lambda_f \theta_f,$$

where the germ $\theta_f$ is biholomorphic at $0$, and $\Lambda_f$ is an invertible diagonal matrix.

**Lemma 3.** For every $h \in P_f(\Omega)$ we have $\theta_h = \theta_f$. 
Proof. Let us consider the germ

\[ \theta = \Lambda_f^{-1} \theta_h f, \]

which is clearly biholomorphic. We have

\[ \theta h = \Lambda_f^{-1} \theta_h f h = \Lambda_f^{-1} \theta_h f h = \Lambda_f \Lambda_h \theta h f = \Lambda_h \Lambda_f^{-1} \theta_h f = \Lambda_h \theta. \]

Using (10), we write the equation \( \theta h = \Lambda_h \theta \) in the coordinate form:

\[ (1/\lambda_{f,i}) \theta_{h,i} f h = (\lambda_h / \lambda_{f,i}) \theta_{h,i} f, \quad i = 1, \ldots, n. \]

By (9) and the definition of \( \xi_i \),

\[ (1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i = (\lambda_h / \lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i, \quad i = 1, \ldots, n, \]

where \( f_i = \xi_i f \xi_i^{-1} \). Using the commutativity relations \( f_i P_i = P_i f_i, \quad h_i P_i = P_i h_i, \) which hold since \( \{p_i\} \subset S_f, \ h \in P_f(\Omega) \), we get

\[ (1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i P_i = (\lambda_h / \lambda_{f,i}) \eta_{h,i} \pi_i f_i P_i, \quad \text{or} \]

\[ (1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i I_i = (\lambda_h / \lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i, \quad i = 1, \ldots, n. \]

This is the same as

\[ ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i) / \pi_i h_i I_i = \lambda_{f,i}, \quad \text{since} \ h_i \ \text{locally preserves the} \ i^{th} \ \text{coordinate axis} \ (h_i P_i = P_i h_i). \]

It is easily seen that

\[ ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)(0) = 0, \]

\[ ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)'(0) = 1. \]

A normalized solution to a Schröder equation is unique, though; thus we have

\[ \eta_{h,i}(\pi_i f_i I_i) / \pi_i h_i I_i = \lambda_{f,i} \eta_{h,i}, \quad \eta_{h,i}(0) = 0, \quad \eta_{h,i}(0) = 1. \]

Using the uniqueness argument again, we obtain \( \eta_{h,i} = \eta_{f,i} \), and hence \( \theta_h = \theta_f \).

The lemma is proved.

According to Lemma 3, the single biholomorphic germ \( \theta_f \) conjugates the subsemigroup \( P_f(\Omega) \) to some subsemigroup \( D_f \) of invertible diagonal matrices in \( D_n \), the set of all \( n \times n \) diagonal matrices with entries in \( \mathbb{C} \). We show that \( D_f \) contains all invertible diagonal matrices with sufficiently small entries. To do this, first we extend \( \theta_f \) to an analytic map on the whole domain \( \Omega \) using the formula

\[ \theta_f = \Lambda_f^{-1} \theta_f f, \]

where \( l \) is chosen so large that \( \text{Cl}\{f'(\Omega)\} \) is contained in a neighborhood of \( 0 \) where \( \theta_f \) is originally defined and biholomorphic; the symbol \( \text{Cl} \) denotes closure. From the procedure of extending \( \theta_f \) to \( \Omega \) we see that it is one-to-one and bounded in the whole domain.

Now, let \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) be a matrix such that \( \text{Cl}\{\Lambda \theta_f(\Omega)\} \subset W \), where \( W \) is a neighborhood of \( 0 \) in \( \mathbb{C}^n \) for which \( \text{Cl}\{\theta_f^{-1} W\} \subset \Omega \). Such a matrix \( \Lambda \) exists since \( \theta_f \) is bounded in \( \Omega \). Consider \( h = \theta_f^{-1} \Lambda \theta_f \), which belongs to \( G_0(\Omega) \). The map \( h \) commutes with \( f \) and all \( p_i \)'s. Indeed, using the formula \( \theta_f f \theta_f^{-1} = \Lambda_f \), we conclude that \( h f = f h \) is equivalent to \( \Lambda F = F \Lambda \), which is a true relation since both matrices \( \Lambda \) and \( \Lambda_f \) are diagonal. The relations \( h p_i = p_i h, \ i = 1, \ldots, n, \) are verified similarly, using the formula \( \theta_f p_i \theta_f^{-1} = P_i \), which follows from the definition of \( \theta_f \).
7. Solving a matrix equation

We proved that for an element \( f \in \text{VG}_0(\Omega) \) there exists a biholomorphic germ \( \theta_f \) conjugating the semigroup \( P_f(\Omega) \) to a subsemigroup \( D_f \subset D_n \), which contains all invertible diagonal matrices with sufficiently small entries.

Let \( f \in \text{VG}_0(\Omega_1) \), and \( g = \varphi f \). Then \( g \in \text{VG}_0(\Omega_2) \), and there is an isomorphism

\[
\Phi : \ S_f \to S_g.
\]

For the mappings \( f \) and \( g \) we have

\[
\theta_f f = \Lambda_f \theta_f, \quad \theta_g g = M_g \theta_g,
\]

where \( \Lambda_f, M_g \) are invertible diagonal matrices.

Let us consider the germ \( L = \theta_g \psi \theta_f^{-1} \). This germ conjugates the semigroups \( D_f, D_g \):

\[
L \Lambda L^{-1} = \theta_g \psi \theta_f^{-1} \Lambda \theta_f \psi^{-1} \theta_g^{-1} = \theta_g \psi \psi^{-1} \theta_g^{-1} = \theta_g j \theta_g^{-1} = M,
\]

where \( h \in P_f, \theta_f h = \Lambda \theta_f; j = \varphi h, \theta_g j = M \theta_g \).

Define \( R(\Lambda) = L \Lambda L^{-1} \). Then \( R : D_f \to D_g \),

\[
R(\Lambda_1 \Lambda_2) = R(\Lambda_1) R(\Lambda_2), \quad \Lambda_1, \Lambda_2 \in D_f.
\]

In what follows, we will identify \( D_n \) with the multiplicative semigroup \( \mathbb{C}^n (D_n \cong \mathbb{C}^n) \) in the obvious way and consider a topology on \( D_n \) induced by the standard topology on \( \mathbb{C}^n \).

We are going to extend \( R \) to an isomorphism of \( D_n \). First, we denote by \( \overline{D}_f, \overline{D}_g \) the closures of \( D_f, D_g \) in \( D_n \), and for \( \Lambda \in \overline{D}_f \) we set

\[
R(\Lambda) = \lim R(\Lambda_k), \quad \Lambda_k \to \Lambda, \quad \Lambda_k \in D_f.
\]

This limit exists and does not depend on the sequence \( \{\Lambda_k\} \), which follows from the fact that \( \psi, \psi^{-1}, \theta_f^{-1}, \theta_g \) are continuous. The map \( R \) is an isomorphism of topological semigroups \( \overline{D}_f \) and \( \overline{D}_g \) (the inverse of \( R \) has a similar representation).

Next, we extend the map \( R \) to \( D_n \) as

\[
R(\Gamma) = R(\Gamma \Lambda) R(\Lambda)^{-1}, \quad \Gamma \in D_n,
\]

where \( \Lambda \in D_f \) is chosen so that \( \Gamma \Lambda \in \overline{D}_f \). This definition does not depend on the choice of \( \Lambda \). Indeed, since all matrices in question are diagonal (hence commute), the relation \( R(\Gamma \Lambda_1) R(\Lambda_1)^{-1} = R(\Gamma \Lambda_2) R(\Lambda_2)^{-1} \) is equivalent to \( R(\Gamma \Lambda_1) R(\Lambda_2) = R(\Gamma \Lambda_2) R(\Lambda_1) \), which holds.

The extended map \( R \) is clearly an isomorphism of \( D_n \) onto itself. Thus we have

\[
R(\Lambda' \Lambda'') = R(\Lambda') R(\Lambda''), \quad \Lambda', \Lambda'' \in D_n.
\]

Injectivity of \( R \) and (11) imply that \( R(\Delta_i) = \Delta_j \) for all \( i \), where \( j = j(i) \) depends on \( i \); \( j(i) \) is a permutation on \( \{1, \ldots, n\} \) (we recall that \( \Delta_i = \text{diag}(0, \ldots, 1, \ldots, 0) \)). This is because \( \{\Delta_i\}_{i=1}^n \) is the only system in \( D_n \) with the following relations:

\[
\Delta_i \neq 0, \quad \Delta_i^2 = \Delta_i, \quad \Delta_i \Delta_j = 0, \quad i \neq j.
\]

Since all matrices \( \Lambda \) and their images \( R(\Lambda) \) are diagonal, we can consider the matrix equation (11) as \( n \) scalar equations:

\[
r_j(\lambda'_1, \lambda'_2, \ldots, \lambda'_n, \lambda''_1, \lambda''_2, \ldots, \lambda''_n) = r_j(\lambda'_1, \ldots, \lambda'_n) r_j(\lambda''_1, \ldots, \lambda''_n), \quad j = 1, \ldots, n,
\]

where \( r_j \) are components of \( R \). If we rewrite the equation \( R(\Delta_i \Lambda) = \Delta_j R(\Lambda) \) in the coordinate form, we see that \( r_j(\lambda_1, \ldots, \lambda_n) = r_j(0, \ldots, \lambda_i, \ldots, 0) = q_j(\lambda_i) \); that
is, each \( r_j \) depends only on one of the \( \lambda_i \)'s. For each \( j \) the corresponding equation in (12) in terms of the \( q_j \)'s becomes
\[
q_j(\lambda_i \lambda_i^j) = q_j(\lambda_i^j)q_j(\lambda_i^j).
\]
This equation has ([4], p. 130) either the constant solution \( q_j(\lambda_i) = 1 \), or
\[
q_j(\lambda_i) = \lambda_i^{\alpha_i - \beta_i}, \quad \alpha_i, \beta_i \in \mathbb{C}, \quad \alpha_i - \beta_i = \pm 1.
\]

Going back to the function \( L \), we have
\[
L_{\text{diag}}(\lambda_1, \ldots, \lambda_n) = \text{diag}(\lambda_{i(1)}^{\alpha_{i(n)}}, \ldots, \lambda_{i(n)}^{\alpha_{i(n)}}, \lambda_{i(n)}^{\beta_{i(n)}}) \cdot L,
\]
where \( i(j) \) is the inverse permutation to \( j(i) \).

Let us choose and fix \((\mu_1, \ldots, \mu_n)\) such that \((1/\mu_1, \ldots, 1/\mu_n)\) belongs to a neighborhood \( W_0 \) of \( 0 \in \mathbb{C}^n \) where \( L \) is defined, and let \( W_1 \) be a neighborhood of \( 0 \in \mathbb{C}^n \) such that \((\mu_1 z_1, \ldots, \mu_n z_n)\) \( \in W_0 \), whenever \((z_1, \ldots, z_n)\) \( \in W_1 \). Then from (13) we have
\[
L(z_1, \ldots, z_n) = L_{\text{diag}}(\mu_1 z_1, \ldots, \mu_n z_n)(1/\mu_1, \ldots, 1/\mu_n)
\]
\[
= \text{diag}\left((\mu_{i(1)} z_{i(1)})^{\alpha_{i(n)}}, \ldots, (\mu_{i(n)} z_{i(n)})^{\alpha_{i(n)}}, \frac{1}{\mu_{i(n)} z_{i(n)}}\right) \cdot \frac{1}{L(1/\mu_1, \ldots, 1/\mu_n)}
\]
\[
= B(z_1^{\alpha_{i(1)}}, \ldots, z_n^{\alpha_{i(n)}}, z_1^{\beta_{i(1)}}, \ldots, z_n^{\beta_{i(n)}}),
\]
where \( B \) is a constant matrix. The last formula is the explicit expression for \( L \).

8. Proving that \( \psi \) is (anti-) biholomorphic

To prove that \( \psi \) is (anti-) biholomorphic is the same as to prove that \( L \) is (anti-) biholomorphic because the relation \( L = \theta_\theta \circ \psi \circ \theta^{-1}_\theta \) holds. We showed that
\[
L(z_1, \ldots, z_n) = B(z_1^{\alpha_{i(1)}}, \ldots, z_n^{\alpha_{i(n)}}, z_1^{\beta_{i(1)}}, \ldots, z_n^{\beta_{i(n)}}), \quad \alpha_i - \beta_i = \pm 1, \quad i = 1, \ldots, n
\]
in a neighborhood \( W_1 \) of \( 0 \). From the representation (14) we see that \( L \) is \( \mathbb{R} \)-differentiable and non-degenerate in \( W_1 \setminus \bigcup_{k=1}^n \{(z_1, \ldots, z_n) : z_k = 0\} \). Since this is true for every point in the domain \( \Omega_1 \), the map \( \psi \) is \( \mathbb{R} \)-differentiable and non-degenerate everywhere, with the possible exception of an analytic set. Let us remove this set from \( \Omega_1 \), as well as its image under \( \psi \) from \( \Omega_2 \). We call the domains obtained in this way \( \Omega' \), \( \Omega'' \). Now the map \( \psi : \Omega' \to \Omega'' \) is \( \mathbb{R} \)-differentiable and non-degenerate everywhere. It is clear that if we prove that \( \psi \) is (anti-) biholomorphic between \( \Omega' \), \( \Omega'' \), then it is (anti-) biholomorphic between \( \Omega_1 \), \( \Omega_2 \) due to a standard continuation argument [11]. So we can think that \( \psi \) is \( \mathbb{R} \)-differentiable and non-degenerate in \( \Omega_1 \) itself. The map \( L \) thus has to be \( \mathbb{R} \)-differentiable and non-degenerate at \( 0 \). However, this is the case if and only if \( \alpha_i + \beta_i = 1 \), \( i = 1, \ldots, n \). Together with the equation \( \alpha_i - \beta_i = \pm 1 \) it gives us that either \( \alpha_i = 1 \), \( \beta_i = 0 \), or \( \alpha_i = 0 \), \( \beta_i = 1 \).

It remains to show that either \( \alpha_i = 1 \) and \( \beta_i = 0 \), or \( \alpha_i = 0 \) and \( \beta_i = 1 \), simultaneously for all \( i \). Suppose, by way of contradiction, that we have \( L(z_1, \ldots, z_n) = B(\ldots, z_{i_1}, \ldots, z_{j_1}, \ldots) \). Then
\[
L^{-1}(w_1, \ldots, w_n) = (\ldots, l_i(w_1, \ldots, w_n), \ldots, l_j(w_1, \ldots, w_n), \ldots),
\]
where \( l_i, l_j \) are linear analytic functions. Let us look at an endomorphism \( f_0 \) of \( \Omega_1 \) in the form
\[
f_0 = \theta^{-1}_\theta(\lambda(\ldots, \theta_{f_1}, \theta_{f_2}, \ldots, \theta_{f,j}, \ldots)\theta_f),
\]
where \( \theta_f, \theta_{f^j} \) is in the \( i^{th} \) place and \( \theta_{f^j} \) in the \( j^{th} \); \( |\lambda| \) is sufficiently small. Using (1) and the definition of \( L \), we have

\[
\theta_{\varphi} f_0 \theta_\varphi^{-1} = \theta_{\psi} f_0 \psi^{-1} \theta_\psi^{-1} = L \theta_f f_0 \theta_f^{-1} L^{-1}.
\]

So,

\[
\theta_{\varphi} f_0 \theta_\varphi^{-1}(w_1, \ldots, w_n)
= B'(l_1(w_1, \ldots, w_n), \ldots, l_j(w_1, \ldots, w_n), \ldots)
\]

for some constant matrix \( B' \). This map, and hence \( \varphi f_0 \), is not analytic, though, in a neighborhood of 0, which is a contradiction. Thus \( L \), and hence \( \psi \), is either analytic or antianalytic in a neighborhood of 0.

Theorem 1 is proved completely.

9. Proof of Theorem 2

Since \( \varphi \) is an epimorphism, it takes constant endomorphisms of \( \Omega_1 \) to constant endomorphisms of \( \Omega_2 \), which follows from (2). Thus we can define a map \( \psi : \Omega_1 \to \Omega_2 \) as in (3). Following the same steps as in verifying (1), we obtain

(15) \( \varphi f \circ \psi = \psi \circ f \), for all \( f \in E(\Omega_1) \).

We will show that (15) implies bijectivity of \( \psi \). The map \( \psi \) is surjective. Indeed, let \( w \in \Omega_2 \) and \( c_w \) be the corresponding constant endomorphism. Since \( \varphi \) is an epimorphism, there exists \( f \in E(\Omega_1) \), such that \( \varphi f = c_w \). If we plug this \( f \) into (15), we get

\( \psi f(z) = w \)

for all \( z \in \Omega_1 \). Thus \( \psi \) is surjective.

To prove that \( \psi \) is injective, we show that for every \( w \in \Omega_2 \), the full preimage of \( w \) under \( \psi \), \( \psi^{-1}(w) \), consists of one point.

Assume for contradiction that \( S_w = \psi^{-1}(w) \) consists of more than one point for some \( w \in \Omega_2 \). The set \( S_w \) cannot be all of \( \Omega_1 \), since \( \psi \) is surjective. For \( z_0 \in \partial S_w \cap \Omega_1 \) we can find \( z_1 \in S_w \) and \( \zeta \notin S_w \) which are arbitrarily close to \( z_0 \). Let \( z_2 \) be a fixed point of \( S_w \) different from \( z_1 \). Consider a homothetic transformation \( h \) such that \( h(z_1) = z_1 \), \( h(z_2) = \zeta \). Since the domain \( \Omega_1 \) is bounded, we can choose points \( z_1 \) and \( \zeta \) sufficiently close to each other so that \( h \) belongs to \( E(\Omega_1) \). Applying (15) to \( h \) we obtain

\[
\varphi h(w) = \varphi h \circ \psi(z_1) = \psi \circ h(z_1) = \psi(z_1) = w;
\]

\[
\varphi h(w) = \varphi h \circ \psi(z_2) = \psi \circ h(z_2) = \psi(\zeta) \neq w.
\]

The contradiction shows injectivity of \( \psi \). Thus we have proved that \( \psi \) is bijective.

According to (15) we have

\[
\varphi f = \psi \circ f \circ \psi^{-1}, \text{ for all } f \in E(\Omega_1),
\]

which implies that \( \varphi \) is an isomorphism.

Theorem 2 is proved.
REFERENCES


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