Math 231/249, Honors problem 2, Spring 2008

1. Let $f(x) = 1/x$, $x > 0$. Find the Taylor series of $f$ about $\alpha = 1$ and find its interval of convergence. Conclude that even though the function $f$ is infinitely differentiable for $x > 0$, the Taylor series does not converge everywhere in this set.

2. Let

$$f(x) = \begin{cases} 
e^{-1/x^2}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Show that $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$. Conclude that the Taylor series of $f$ about $\alpha = 0$ is a convergent series that does not converge to $f$.

**Hint:** Use the fact that

$$\lim_{h \to 0} \frac{e^{-1/h^2}}{h^n} = 0$$

for all $n$. 
1. $f(x) = \frac{1}{x}, \quad x > 0$

Taylor series of $f$ about $a = 1$?

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{k=0}^{\infty} (1-x)^k = \sum_{k=0}^{\infty} (-1)^k (x-1)^k$$

- geometric series

Converges absolutely for $|x-1| < 1$;

diverges for $|x-1| \geq 1$.

$x > 0$: converges abs. for $0 < x < 2$

diverges for $x \geq 2$.

Though the function $f$ is defined and infinitely differentiable for all $x > 0$. 
\( f(x) = \begin{cases} e^{-\frac{1}{2}x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \)

Taylor series of \( f \) about \( x = 0 \)?

\( f(0) = 0 \).

\( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-\frac{1}{2}x^2}}{x} = 0 \quad \text{< Hint>}. \)

\( f''(0) = \lim_{x \to 0} \frac{f(x) - f'(0)}{x - 0} \quad \text{< } \)

\( f'(x) = \begin{cases} \frac{2}{x^3} e^{-\frac{1}{2}x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \)

\( \lim_{x \to 0} \frac{2}{x^3} e^{-\frac{1}{2}x^2} = 0 \quad \text{< Hint>}

In general, \( n \geq 0 \)

\( f^{(n)}(0) = \lim_{x \to 0} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x - 0} \),

\( f^{(n+1)}(x) = P \left( \frac{1}{x} \right) e^{-\frac{1}{2}x^2} \),

\( P \) - polynomial.

\( \text{< Hint> } f^{(n)}(0) = 0 \) \( \forall \) \( n \geq 0 \).

Taylor series: \( 0 = 0 + 0 + \ldots \)

converges to \( f \) only at \( x = 0 \).

Though \( f \) is infinitely differentiable for all \( x \).