ON THE MAXIMAL INEQUALITIES FOR MARTINGALES INVOLVING TWO FUNCTIONS

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Abstract. Let \(\Phi(t)\) and \(\Psi(t)\) be nonnegative convex functions, and let \(\varphi\) and \(\psi\) be the right continuous derivatives of \(\Phi\) and \(\Psi\), respectively. In this paper, we prove the equivalence of the following three conditions: (i) \(\|f^*\|_\varphi \leq c\|f\|_\varphi\), (ii) \(L^\Phi \subseteq H^\Psi\), and (iii) \(\int_{s_0}^{t} \frac{d\varphi(s)}{s} ds \leq c\psi(\varepsilon t), \forall t > s_0\), where \(L^\Phi\) and \(H^\Psi\) are the Orlicz martingale spaces. As a corollary, we get a sufficient and necessary condition under which the extension of Doob's inequality holds. We also discuss the converse inequalities.

1. Introduction

Let \(\Phi\) be a nonnegative convex continuous function on \([0, \infty)\) with \(\Phi(0) = 0\) and \(\lim_{t \to \infty} \Phi(t) = \infty\), and \((\Omega, \mathcal{F}, \mu)\) be a complete probability space. We denote by \(\mathcal{M}\) the set of all \(\mathcal{F}\)-measurable functions and put

\[ L^\Phi(\Omega) = \{f \in \mathcal{M}, \exists \varepsilon > 0, E\Phi(\varepsilon |f|) < \infty\} \]

where \(E\) stands for the expectation with respect to \(\mu\). \(L^\Phi(\Omega)\) is the so-called Orlicz space (see Rao [11]). Define \(\|f\|_\Phi = \inf\{k > 0, E\Phi(\frac{|f|}{k}) \leq 1\}\). Let \(\mathcal{F}_n (n \geq 1)\) be a nondecreasing sequence of complete sub-\(\sigma\)-fields and \(\mathcal{F} = \bigvee_{n=0}^{\infty} \mathcal{F}_n\). Denote the maximal function and the \(\Phi\)-norm of a submartingale \(f = (f_n)_{n \geq 0}\) adapted to \(\mathcal{F}_n (n \geq 1)\) respectively by

\[ f^*(\omega) = \sup_{n \geq 0} |f_n(\omega)|, \quad \|f\|_\Phi = \sup_{n \geq 0} \|f_n\|_\Phi. \]

It is well known that if \(\Phi\) is a strictly convex function on \([0, \infty)\), i.e. \(q_\Phi = \inf_{t > 0} \frac{t\varphi(t)}{\Phi(t)} > 1\) (where \(\varphi\) is the right continuous derivative of \(\Phi\)), the extension of the classical Doob's inequality

\[ \|f^*\|_\Phi \leq c \sup_{n \geq 0} \|f_n\|_\Phi \]

holds for every martingale or nonnegative submartingale \(f = (f_n)_{n \geq 0}\); here \(c\) is a constant depending only on \(\Phi\). When \(\Phi\) is not strictly convex, the situation

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is rather different. To see this, recall Doob’s inequality in the case of \( p = 1 \) and Imkeller’s inequality [4]
\[
E f^* \leq \frac{e}{e-1} (1 + \sup_{n \geq 0} E |f_n| \log^+ |f_n|), \quad \|f^*\|_1 \leq c \sup_{n \geq 0} \|f_n\|_\Psi,
\]
where \( \Psi = t \log^+ t \), which hold for every martingale or nonnegative submartingale \( f \). It means that \( f^* \in L^1 \) if \( f \in L \log^+ L \).

In [3–7], Kita considered some conditions about two such functions \( \Phi \) and \( \Psi \), for which the inequality
\[
(1) \quad \int_{\mathbb{R}^n} \Phi(f^*(x))dx \leq c \int_{\mathbb{R}^n} \Psi(c|f(x)|)dx
\]
holds for every function \( f \in L^\Psi \) defined in \( \mathbb{R}^n \), where \( f^* \) is the Hardy-Littlewood maximal function of \( f \). In particular, he observed that condition (1) holds for every function \( f \in L^\Psi \) if and only if \( \int_0^t \frac{\phi(s)}{s} ds \leq c \psi(ct), \forall t > 0 \), where \( \phi, \psi \) stand for the right continuous derivatives of \( \Phi \) and \( \Psi \), respectively and \( c > 0 \) is some constant. He also considered the condition for which the converse inequality holds. Moreover, the authors noticed that D. Gilat [2] had obtained a sharp inequality that gives a comparison of the \( L_p \) norm of a martingale to the \( L_1 \) norm of its maximal function.

In this paper, we consider the Orlicz spaces of martingales and prove the equivalence of the following three conditions: (i) \( \|f^*\|_\Phi \leq c \|f\|_\Psi \) for every martingale or nonnegative submartingale \( f \), (ii) \( L^\Psi \subseteq H^\Phi \), (iii) \( \int_{s_0}^\infty \frac{\phi(s)}{s} ds \leq c \psi(ct), \forall t > s_0 \), where \( s_0, c \) are some positive constants and the Orlicz martingale spaces \( L^\Phi \) and \( H^\Phi \) are defined as
\[
\begin{align*}
L^\Phi &= \{ f = (f_n)_{n \geq 0} : \|f\|_\Phi < \infty \}, \\
H^\Phi &= \{ f = (f_n)_{n \geq 0} : \|f\|_{H^\Phi} = \|f^*\|_\Phi < \infty \}.
\end{align*}
\]
As a corollary, a sufficient and necessary condition under which the extension of Doob’s inequality holds is obtained. We also consider the converse inequalities for the regular martingales.

In this paper \(|A|\) always means the measure of set \( A \) with respect to \( \mu \).

2. THE MAXIMAL INEQUALITIES

Lemma 1. Let \( f = (f_n)_{n \geq 0} \) be a nonnegative submartingale. Then
\[
(2) \quad |\{f_n^* > t\}| \leq \frac{2}{t} \int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}|d\lambda, \quad \forall t > 0, \ n \in \mathbb{N},
\]
and
\[
(3) \quad \int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}|d\lambda \leq \int_{\{f_n > \frac{t}{2}\}} f_n d\mu.
\]

Proof. For \( t > 0 \), let \( h_n = (f_n \vee \frac{t}{2}) - \frac{t}{2} \) for every \( n \in \mathbb{N} \). It is easy to see that \( h = (h_n)_{n \geq 0} \) is a nonnegative submartingale. By using Kolmogorov’s inequality and Fubini’s theorem we have
\[
|\{f_n^* > t\}| = |\{h_n^* > \frac{t}{2}\}| \leq \frac{2}{t} \int \Omega h_n d\mu = \frac{2}{t} \int \Omega (f_n \vee \frac{t}{2}) d\mu - 1
\]
\[
= \frac{2}{t} \int_0^\infty |\{f_n \vee \frac{t}{2} > \lambda\}|d\lambda - 1 = \frac{2}{t} \int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}|d\lambda
\]
which implies (2). From this equation we get
\[ \frac{2}{t} \int_{t}^{\infty} |\{f_n > \lambda\}|d\lambda = \frac{2}{t} \int_{\Omega} \lambda \frac{t}{2} d\mu - 1 \leq \frac{2}{t} \int_{\{f_n > \frac{t}{2}\}} f_n d\mu \]
which implies (3).

**Theorem 1.** Suppose that \( \Phi, \Psi \) are nondecreasing convex functions defined on \([0, \infty)\) with \( \Phi(0) = \Psi(0) = 0 \). Then the following statements are equivalent:

(i) There exist \( c_0, c_1 > 0 \) such that
\[ \int_{s_0}^{t} \frac{\varphi(s)}{s} ds \leq c_1 \varphi(c_1 t), \quad \forall t > s_0. \]

(ii) There exists \( c_4 > 0 \) such that
\[ \| f^* \| \Phi \leq c_4 \| f \| \Psi \]
for every martingale or nonnegative submartingale \( f = (f_n)_{n \geq 0} \).

(iii) \( L^\Psi \subseteq H^\Phi \).

**Proof.** (i)\(\Rightarrow\)(ii). We only prove (5) for nonnegative submartingales. By using Lemma 1 and Fubini’s theorem we get
\[ E\Phi(f_n^*) = \int_{0}^{\infty} |\{f_n^* > t\}|d\Phi(t) \]
\[ \leq \int_{s_0}^{\infty} |\{2f_n^* > 2t\}|d\Phi(t) + \Phi(s_0) \]
\[ \leq \int_{s_0}^{\infty} \frac{1}{t} \int_{t}^{\infty} |\{2f_n > \lambda\}|d\lambda d\Phi(t) + \Phi(s_0) \]
\[ = \int_{s_0}^{\infty} |\{2f_n > \lambda\}| \int_{s_0}^{\infty} \frac{1}{t} d\Phi(t) d\lambda + \Phi(s_0) \]
\[ \leq E\Psi(2c_1 f_n) + \Phi(s_0). \]
Without loss of generality, we assume that \( \|2c_1 f\|_\Psi = 1 \). Then \( E\Phi(f^*) \leq 1 + \Phi(s_0) \) and \( E\Phi(\frac{f^*}{1 + \Phi(s_0)}) \leq 1 \). Hence
\[ \| f^* \|_\Phi \leq 1 + \Phi(s_0) \leq c_4 \| f \|_\Psi, \]
where \( c_4 = 2(1 + \Phi(s_0))c_1 \), as desired.

(ii)\(\Rightarrow\)(iii). First we prove that (5) holds for every nonnegative martingale \( f = (f_n)_{n \geq 0} \) with \( \|f\|_\Psi = 1 \). Assume it is not the case. Then for every \( n \), choose a nonnegative martingale \( f^{(n)} = (f_{nk})_{k \geq 0} \) defined on \((\Omega_n, \mathcal{F}_n, \mathcal{F}_nk, \mu_n)\) satisfying \( \|f^{(n)}\|_\Psi = 1 \), \( \| f^{(n)} \|_\Phi = 4^n \). Consider the product space \((\Omega, \mathcal{F}, \mathcal{F}_k, \mu) = \prod_{n=1}^{\infty} (\Omega_n, \mathcal{F}_n, \mathcal{F}_nk, \mu_n)\), where \( \mathcal{F}_k = \prod_{n=1}^{\infty} \mathcal{F}_nk \) and take \( h^{(n)} = (h_{nk})_{k \geq 1} \), where \( h_{nk} = \chi_{\Omega_1} \cdots \chi_{\Omega_{n-1}} \times f_{nk} \times \chi_{\Omega_{n+1}} \times \cdots \). It is easy to see that \( h^{(n)} \) is a martingale with \( \| h^{(n)} \|_\Psi = 1 \) and \( \| h^{(n)} \|_\Phi > 4^n \). Let \( g_k(\omega, t) = \sum_{n=1}^{\infty} \frac{h_{nk}(\omega)}{h_{nk}(\omega)} \). Then \( \| g \|_\Psi \leq \sum_{n=1}^{\infty} \| \frac{h_{nk}}{h_{nk}(\omega)} \|_\Psi = 1 \) and \( \| \frac{h_{nk}}{h_{nk}(\omega)} \|_\Phi \geq \| \frac{h^{(n)}(\omega)}{h_{nk}(\omega)} \|_\Phi > 1, \forall n > 0 \). This contradicts the fact that \( g \in L^\Psi \subseteq H^\Phi \).

To prove (i), let \( A_k = (0, \frac{1}{2^k}) \), \( \mathcal{F}_k = \sigma\{A_0, A_1, A_2, \cdots, A_k\} \), \( f = t\chi_{A_n}, f = (E(f | \mathcal{F}_k))_{k \geq 0} \), where \( k, n \geq 0, t = \Psi^{-1}(2^n) \). Then \( f \) is a finite dyadic martingale.
on $(0, 1]$ with \( \frac{t}{s} \leq f^*(\omega) \leq t \). It is clear that \( \|f\|_\psi = 1 \) and then \( \|f^*\|_\Phi \leq c_4 \|f\|_\psi = c_4 \). Thus \( E\Phi\left( \frac{f^*}{c_4} \right) \leq 1 \), i.e.

\[
\int_0^\infty \mathbb{1}\{f^* > cs\}\varphi(s)ds \leq 1.
\]

Notice that \( \frac{1}{s} \leq c_4 \frac{2^{n-k}}{t} \) and \( \{f^* > cs\} = \frac{1}{2^{n-k}} \), when \( s \in (\frac{t}{c_42^{n-k}}, \frac{t}{c_42^{n-k+1}}) \) \( (0 \leq k \leq n - 1) \). We have

\[
\frac{1}{s} \leq c_4 \frac{2^{n+1}}{t} \cdot \frac{1}{2^{k+1}} = c_4 \frac{2^{n+1}}{t} \cdot \mathbb{1}\{f^* > cs\}, \quad \forall s \in (\frac{t}{c_42^n}, \frac{t}{c_4}).
\]

From (7) we get

\[
\int_{\frac{t}{c_42^n}}^{\frac{t}{c_4}} \frac{\varphi(s)}{s}ds \leq c_4 \frac{2^{n+1}}{t} \int_0^\infty \mathbb{1}\{f^* > cs\}\varphi(s)ds \leq c_4 \frac{2^{n+1}}{t} \mathbb{1}\{f^* > cs\} \varphi(s)ds \leq 2c_4\psi(t),
\]

By the convexity of \( \psi \), we have \( \frac{t}{c_42^n} \leq \frac{\psi^{-1}(1)}{c_4} = s_0 \), \( \forall n \geq 0 \), and then \( \int_{s_0}^{\frac{t}{c_4}} \frac{\varphi(s)}{s}ds \leq 2c_4\psi(t) \),

\[
\int_{s_0}^{s_0} \frac{\varphi(s)}{s}ds \leq 2c_4\psi(\psi^{-1}(2^{n+1})) \leq 2c_4\psi(2c_4u), \quad \forall u \in \left(\frac{1}{c_4}\psi^{-1}(2^n), \frac{1}{c_4}\psi^{-1}(2^{n+1})\right], \, n \geq 0.
\]

That is,

\[
\int_{s_0}^{t} \frac{\varphi(s)}{s}ds \leq c_1\psi(c_1t), \quad \forall t \geq s_0,
\]

where \( c_1 = 2c_4 \). This completes the proof.

Letting \( \Phi = \psi \) in Theorem 1, we get

**Corollary 1.** Suppose that \( \Phi \) is a nondecreasing convex function defined on \([0, \infty)\) with \( \Phi(0) = 0 \). Then

\[
\|f^*\|_\Phi \leq c \sup_n \|f_n\|_\Phi
\]

holds for every martingale or nonnegative submartingale \( f = (f_n)_{n \geq 0} \) if and only if there exist \( s_0, c_1 > 0 \) such that

\[
\int_{s_0}^{t} \frac{\varphi(s)}{s}ds \leq c_1\varphi(c_1t), \quad \forall t \geq s_0.
\]

We will prove in Section 3 that (8) is equivalent to the condition \( \lim_{t \to \infty} \frac{t\varphi(t)}{\Phi(t)} > 1 \) when \( \Phi \in \Delta_2 \) (i.e. there exist \( c, t_0 > 0 \) such that \( \Phi(2t) \leq c\Phi(t), \forall t > t_0 \)).

Stein [12] proved that if \( f \) is a function supported in a ball \( B \subseteq \mathbb{R}^n \), then \( f^* \in L^1(B) \) implies \( f \in L\log^+L \). Kita [7] proved some similar results on the Orlicz function spaces. The following theorem shows that the situation for martingales is different, i.e. for martingale space, it can happen that \( H^1 \subseteq L\log^+L \).

**Theorem 2.** Suppose that \( \Phi, \Psi \) are nondecreasing convex functions defined on \([0, \infty)\) with \( \Phi(0) = \Psi(0) = 0 \). Then the following statements are equivalent:

(i) There exists \( c > 0 \) such that \( \lim_{t \to \infty} \frac{\varphi(ct)}{\Phi(t)} > 0 \).

(ii) \( \|f^*\|_\Psi \leq L^\Phi \).
(iii) $H_+^\Phi \subseteq L_+^\Psi$, where $H_+^\Phi$ and $L_+^\Psi$ are the sets of all positive members of $H^\Phi$, $L^\Psi$, respectively.

Proof. (i)$\Rightarrow$(ii). The condition $\lim_{t \to \infty} \frac{\Phi(ct)}{\Psi(t)} > 0$ means that there exist $c_1 > 0$ and $n_0 \in \mathbb{N}$ such that $\Phi(c_1 t) > \Psi(t)$, $\forall t > n_0$. For $f \in H_+^\Phi$, there exists $k > 0$ such that $E\Phi(\frac{f}{k}) < M < \infty$, and then

$$E\Psi(\frac{f}{c_1 k}) < \Psi(t_0) + E\{\frac{f}{c_1 k^2 k^2 n_0}\} \Phi(\frac{f}{k}) < \Psi(t_0) + M < \infty;$$

hence $f \in L_+^\Psi$. It shows that $H_+^\Phi \subseteq L_+^\Psi$.

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i). When $H_+^\Phi \subseteq L_+^\Psi$, it follows from the proof of (iii)$\Rightarrow$(i) in Theorem 1 that there exists $c > 0$ such that

$$\|f\|_\Psi \leq c \|f^*\|_\Phi$$

for every nonnegative martingale $f = (f_n)_{n \geq 0} \in H_+^\Phi$ defined on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mu)$. Now, for every such a martingale $f$ with $E\Psi(f) \geq 1$, define $g = (g_m)_{m \geq 0}$ on the product space $(\Omega \times (0,1], \mathcal{F} \times \sigma\{(0,\alpha]\}, \mu \times \nu)$:

$$g_m(\omega, t) := f_m(\omega)\chi_{(0,\alpha]}(t), \quad \forall t \in (0,1],$$

where $\nu$ is Lebesgue measure on $(0,1]$ and $\alpha = \frac{1}{E\Psi(f)}$. Denote by $\tilde{E}$ the expectation with respect to $\Omega \times (0,1]$. Then $\tilde{E}\Psi(g) = \alpha E\Psi(f) = 1$ and $\|cg^*\|_\Phi \geq \|g\|_\Psi = 1$. It follows that $\tilde{E}\Phi(cg^*) \geq 1$ and then

$$E\Phi(cf^*) \geq E\Psi(f).$$

Let $t \geq \Psi^{-1}(1)$ and $f \equiv t$. Then $E\Psi(f) \geq 1$. From (9) we have $\Phi(ct) \geq \Psi(t)$, $\forall t \geq \Psi^{-1}(1)$. This implies that $\lim_{t \to \infty} \frac{\Phi(ct)}{\Psi(t)} > 0$. \hfill $\square$

Corollary 2. Suppose that $\Phi, \Psi$ are nondecreasing convex functions on $[0, \infty)$ with $\Phi(0) = \Psi(0) = 0$. Then $H^\Phi = L^\Psi$ if and only if $H^\Phi = L^\Psi$.

Proof. We only prove the sufficiency, and the necessity can be obtained from the "symmetry". Indeed, from $H^\Phi = L^\Psi$ and Theorems 1 and 2, there exist $s_0, c_1 > 0$ such that

$$\int_{s_0}^t \frac{\varphi(s)}{s} ds \leq c_1 \psi(c_1 t), \quad \Phi(c_1 t) > \Psi(t), \quad \forall t > s_0.$$

A simple computation shows that $c_1 \varphi(c_1 t) > \frac{2}{5} \psi(\frac{1}{2})$, $\forall t > s_0$, and

$$\int_{c_1 s_0}^t \frac{\psi(s)}{s} ds \leq \int_{c_1 s_0}^t \frac{\varphi(s)}{s} ds \leq c_1 \psi(c_1 t) \leq 2c_1^2 \varphi(2c_1^2 t), \quad \forall t > c_1 s_0;$$

therefore

$$\int_{s_1}^t \frac{\psi(s)}{s} ds \leq c \varphi(ct), \quad \forall t > s_1,$$

where $s_1 = \frac{s_0}{2}$. Thus $H_+^\Phi \subseteq L_+^\Psi$. The converse relation comes from $H^\Phi = L^\Psi$ directly, and hence $H^\Phi = L^\Phi$ is proved.
Now we consider regular martingales. Recall that an increasing sequence $(\mathcal{F}_n)_{n \geq 0}$ of sub-σ-fields $(\mathcal{F}_0 = \{\emptyset, \Omega\})$ is said to be regular if there exists a $d(\mathcal{F}_n) > 0$ such that

$$
\chi(F) \leq d(\mathcal{F}_n)E(\chi(F)|\mathcal{F}_{n-1}), \quad \forall F \in \mathcal{F}_n, \quad n \in \mathbb{N}.
$$

We say that a probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mu)$ (a submartingale $f = (f_n, \mathcal{F}_n)_{n \geq 0}$) is regular if $(\mathcal{F}_n)_{n \geq 0}$ is regular.

**Lemma 2** ([S]). Let $f = (f_n)_{n \geq 0}$ be a regular nonnegative martingale with the constant $d(\mathcal{F}_n)$. Then

$$
\int_{\{f_n > \lambda\}} f_n d\mu \leq d(\mathcal{F}_n) \lambda \|f_n^* > \lambda\|, \quad \forall \lambda \geq \|f_0\|_{\infty}.
$$

**Theorem 3.** Suppose that $\Phi, \Psi$ are nondecreasing convex functions defined on $[0, \infty)$ with $\Phi(0) = \Psi(0) = 0$. Then the following statements are equivalent:

(i) There exist $s_0, c_1 > 0$ such that

$$
\int_{s_0}^{t} \frac{\varphi(s)}{s} ds \geq c_1 \psi(c_1 t), \quad \forall t > s_0.
$$

(ii) $f \in H^\Phi$ implies that $f \in L^\Psi$ for every nonnegative regular martingale $f = (f_n)_{n \geq 0}$.

(iii) There exists $c_2 > 0$ such that

$$
c_2 \|f^*\|_\Phi \geq \sup_n \|f_n\|_\Psi
$$

for every nonnegative regular martingale $f = (f_n)_{n \geq 0}$, where $c_2$ depends only on $d(\mathcal{F}_n)$.

**Proof.** (i)$\Rightarrow$(ii). For a nonnegative regular martingale $f \in H^\Phi$, choose $c > s_0$ such that $\|f\|_1 \leq c$. By using Fubini’s theorem and Lemmas 1 and 2 we get

$$
E\Psi(c_1 |f|) = \int_0^\infty |\{f > s\}| c_1 \psi(c_1 s) ds
\leq \int_{s_0}^\infty |\{f > s\}| \left(\int_{s_0}^{s} \frac{\varphi(t)}{t} dt + \psi(c_1 c)\right) ds
\leq \int_1^\infty \frac{\varphi(t)}{t} \left(\int_{t}^{\infty} |\{f > s\}| ds\right) dt + \psi(c_1 c)
\leq \int_1^\infty \frac{\varphi(t)}{t} \left(\int_{|f| > t} |\{f > s\}| ds\right) dt + \psi(c_1 c)
\leq \int_1^\infty \frac{\varphi(t)}{t} d(\mathcal{F}_n)t |f^* > t| dt + \psi(c_1 c)
\leq d(\mathcal{F}_n)E\Phi(f^*) + \psi(c_1 c) < \infty.
$$

Hence $f \in L^\Psi$.

(ii)$\Rightarrow$(iii). Without loss of generality, we assume that $\|f^*\|_\Phi = 1$. If the assertion of (iii) is not true, we can choose a sequence of nonnegative martingales $f^{(n)} = (f_{nm})_{m \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathcal{F}, \mu)$ such that $4^n \|f^{(n)*}\|_\Phi < \|f^{(n)}\|_\Psi$. Let $g_k(\omega, t) = \sum_{n=1}^{\infty} \frac{f^{(n)}(\omega, t)}{2^n}$. Then $\|g^*\|_\Phi \leq \sum_{n=1}^{\infty} \left\|\frac{f^{(n)}*}{2^n}\right\|_\Phi = 1$ and $\|g\|_\Psi \geq \|\frac{f^{(n)}*}{2^n}\|_\Psi > 1$, $\forall n > 0$, a contradiction with $g \in H^\Phi \subseteq L^\Psi$. 


Thus for every (10), the convexity of \( \Psi \) and the fact

\[ \Phi(2f^*) > 1 = \Phi(f). \]

Let \( c_4 = (2f(1) \vee 2)c_2 \). By the convexity of \( \Phi \) we have that \( E\Phi(c_4f^*) > 2f(1) \vee 2 \). Notice that \( \int_0^1 |c_4f^* > s)|\varphi(s)ds \leq \Phi(1) \),

and then

\[ \int_1^\infty |c_4f^* > s)|\varphi(s)ds > \frac{1}{2} \int_0^\infty |c_4f^* > s)|\varphi(s)ds > 1. \]  

When \( s \in (\frac{c_4t}{2^{n+k}}, \frac{c_4t}{2^{n+k-1}}) \)(0 \leq k \leq n - 1), we have that

\[ \frac{1}{s} \geq \frac{2^n}{c_4t} \]  

and when \( s \in (0, \frac{c_4t}{2^n}) \), we have \( \frac{1}{s} \geq \frac{2^n}{c_4t} \) \(|c_4f^* > s)|, \forall s < c_4t \). From (10), the convexity of \( \Psi \) and the fact \( f^* \leq t \) we get

\[ \int_1^{c_4t} \frac{\varphi(s)}{s}ds \geq \frac{2^n}{c_4t} \int_1^{c_4t} |c_4f^* > s)|\varphi(s)ds \geq \frac{2^n}{c_4t} \geq \frac{1}{2c_4} \psi(t^2). \]

Thus for every \( n \in N \), when \( u \in (c_4\Psi^{-1}(2^n), \ c_4\Psi^{-1}(2^{n+1})) \) we have

\[ \int_1^{u^2} \frac{\varphi(s)}{s}ds \geq \int_1^{c_4\Psi^{-1}(2^n)} \frac{\varphi(s)}{s}ds \geq \frac{1}{2c_4} \psi(\frac{1}{24}) \geq \frac{1}{2c_4} \psi(\frac{u}{4c_4}); \]

hence

\[ \int_1^t \frac{\varphi(s)}{s}ds \geq c_1 \psi(c_1t), \ \forall t > c_4\Psi^{-1}(2) \]

where \( c_1 = \frac{1}{4c_4} \), which implies (i) and the proof is complete.

**Remark.** There is an example which shows that Theorem 3 is not true if we replace the nonnegative regular martingale in (ii) and (iii) by a regular martingale. To see this, let \( \Phi(t) = t, \Psi(t) = t \log^+ t \), then \( \Phi, \Psi \) satisfy (i). Consider the martingale \( f = (E(4^n\chi_{(0, \frac{1}{4^n})}) - 4^n\chi_{(\frac{1}{2^n}, \frac{1}{4^n}) |F_k})_{k \geq 0} \) on \((0, 1)\), where \( F_k = \sigma\{(0, \frac{1}{4^n}), \ 1 \leq k \leq n \} \). It is easy to check that \( \|f\|_\Psi > \sqrt{n}\|f^*\|_\Psi \), when \( n \) is big enough. The proof of (ii) \( \implies \) (iii) shows that \( H^\Psi \subseteq L^\Psi \).

**Corollary 3.** Suppose that \( \Phi, \Psi \) are nondecreasing convex functions defined on \([0, \infty)\) with \( \Phi(0) = \Psi(0) = 0 \). Then the following statements are equivalent:

(i) There exist \( s_0, c_1 > 0 \) such that \( \int_1^t \frac{\varphi(s)}{s}ds \geq c_1 \psi(c_1t), \ \forall t > s_0 \).

(ii) There exists \( c > 0 \) such that

\[ E\psi(f) \leq c + d_{\bar{\Psi}}(f_n)E\Phi(c^f) \]

for every nonnegative regular martingale \( f = (f_n)_{n \geq 0} \).

**Proof.** We only prove (i) \( \implies \) (ii). (ii) \( \implies \) (i) can be obtained from Theorem 3. From the proof of Theorem 3, we have \( E\psi(f) \leq c + \bar{d}_{\bar{\Psi}}(f_n)E\Phi(c^f) \) for every nonnegative regular martingale \( f \) with \( \|f\|_1 \leq 1 \), where \( c = \Psi(c_1) \). Now for a nonnegative martingale \( f = (f_n)_{n \geq 0} \) adapted to \((\mathcal{F}_n)_{n \geq 0}\) with \( \|f\|_1 = 2^m, \ m \in N \), consider the martingale \( g = (g_n)_{n \geq 0} \) defined on \((\Omega, \mathcal{F}_n, \mu) = (\Omega \times (0, 1), \mathcal{F}_n \times \sigma\{(0, \frac{1}{4^n})\}, \ 1 \leq k \leq n \}, \mu \times \nu \), \( g_n(\omega, t) = E(f\chi(0, \frac{1}{4^n}) |\mathcal{F}_n) \), where \( \nu \) is Lebesque measure on \((0, 1)\). Then \( \|g\|_1 = 1 \) and

\[ E\psi(c_1g) \leq c + \bar{d}_{\bar{\Psi}}(f_n)E\Phi(g^*) = c + 2d_{\bar{\Psi}}(f_n)E\Phi(g^*). \]
Notice that $E\Phi(g^*) \leq \frac{1}{2n}E\Phi(f^*) + \sum_{k=1}^{m} \left( \frac{1}{2^{k}} \right) E\Phi(f^*_k) \leq E\Phi(f^*)$; we get
\begin{equation}
(11) \quad \frac{1}{2n} E\Psi(c_1 f) \leq c + 2d_1 E\Phi(f^*).
\end{equation}
For any $f$ with $\|f\|_1 > 1$, choose $c_0, m$ such that $2^{m-1} < \|f\|_1 \leq 2^m, c_0 = \frac{2^m}{\|f\|_1} < 2$.
From (11) we have
\begin{align*}
E\Psi(c_1 f) & \leq E\Psi(c_1 c_0 f) \leq c_0 \|f\|_1 + 2d_1 \|f\|_1 E\Phi(c_0 f^*) \\
& \leq 2c_0 \|f\|_1 + 4d_1 \|f\|_1 E\Phi(2f^*).
\end{align*}
Thus for every nonnegative regular martingale $f$, we get
\begin{equation}
E\Psi(c_1 f) \leq 2c_0 (\|f\|_1 + 1) + 4d_1 \|f\|_1 (\|f\|_1 + 1) E\Phi(2f^*)
\end{equation}
and
\begin{equation}
E\Psi(f) \leq \frac{2c_0}{c_1} (\|f\|_1 + 1) + \frac{4c_1}{d} \|f\|_1 (\|f\|_1 + 1) E\Phi(\frac{2f^*}{c_1}).
\end{equation}
Choose $c_2 > c_1$ such that $\frac{2c_0}{c_1} > \frac{4c_1}{d}, \forall t \geq c_2$; we obtain
\begin{equation}
E\Psi(f) \leq c_3 + c_4 d_1 \|f\|_1 E\Phi(c_1 f^*)
\end{equation}
where $c_3 = \frac{2c_0}{c_1} c_2, c_4 = \frac{8}{c_1}, c_5 = \frac{2}{c_1}$. Hence the assertion in (ii) is true.

3. SOME EQUIVALENT CONDITIONS AND EXAMPLES

Corollary 1 shows that condition (8) is sufficient and necessary for the extension of Doob’s inequality. In the following, we discuss the relationship between (8) and the condition $q_0 > 1$ when $\Phi \in \Delta_2$ (i.e. there exist $c, t_0 > 0$ such that $\Phi(2t) \leq c\Phi(t), \forall t > t_0$).

**Lemma 3.** If $\Phi$ is a nondecreasing convex function on $[0, \infty)$ with $\Phi(0) = 0$, and there exists $c > 0$ such that $\Phi(2t) \leq c\Phi(t), \forall t > 0$, then $q_0 > 1$ if and only if there exists $c_1 > 0$ such that $\int_0^t \frac{\varphi(s)}{s} ds \leq c_1 \varphi(c_1 t), \forall t > 0$.

**Proof.** First we prove the sufficiency. Notice that $\frac{\varphi(s)}{s} \leq \frac{1}{q_0} \frac{\varphi(s)}{s}$; we get
\begin{equation}
\int_0^t \frac{\varphi(s)}{s} ds \leq \frac{q_0}{q_0 - 1} \int_0^t \frac{\varphi(s)}{s^2} ds \leq \frac{q_0}{q_0 - 1} \frac{\Phi(t)}{t} \leq \frac{q_0}{q_0 - 1} \varphi(t).
\end{equation}

Next we prove the necessity. Notice that the condition $\int_0^t \frac{\varphi(s)}{s} ds \leq c_1 \varphi(c_1 t)$ implies $\varphi(s) \leq 0$ (as $s \to 0$) and $\int_0^t \frac{\varphi(s)}{s} ds < \infty, \forall t > 0$. Denote
\begin{equation}
ak = 2^k \varphi(2^k) - \Phi(2^k) - [2^{k-1} \varphi(2^{k-1}) - \Phi(2^{k-1})] \quad (-\infty < k < +\infty);
\end{equation}
we have
\begin{equation}
2^{k-1} [\varphi(2^k) - \varphi(2^{k-1})] \leq a_k \leq 2^k [\varphi(2^k) - \varphi(2^{k-1})]
\end{equation}
and
\begin{equation}
\sum_{k=-\infty}^{m} 2^{-k} a_k \leq \varphi(2^m) - \varphi(0) \leq \sum_{k=-\infty}^{m} 2^{-k+1} a_k.
\end{equation}
Therefore,
\begin{equation}
\Phi(2^m) \leq \sum_{k=-\infty}^{m-1} 2^k \varphi(2^{k+1}) \leq \sum_{k=-\infty}^{m-1} 2^k \sum_{i=-\infty}^{k+1} 2^{-i+1} a_i \leq \sum_{i=-\infty}^{m} 2^{m-i+1} a_i.
\end{equation}
On the other hand, we have

\[
\int_0^m \frac{\varphi(s)}{s} ds \geq \sum_{k=0}^{m-1} 2^k \varphi\left(\frac{2^k}{2k+1}\right) \geq \frac{1}{2} \sum_{k=0}^{m-1} \sum_{i=-\infty}^k 2^{-i}a_i = \frac{1}{2} \sum_{i=-\infty}^{m-1} 2^{-i}(m-i)a_i.
\]

Notice that \(\inf_{t>0} \frac{t^p-\Phi(t)}{\Phi(t)} = 0\) if \(\Phi = 1\). From (12) and (15) we get that \(\forall j > 1, \exists t_j \in (2^{n_j}, 2^{n_j+1}]\) such that \(\Phi\left(\frac{t_j}{2}\right) > 0\) and

\[
(\sum_{k=-\infty}^{n_j} 2^{n_j-k+1}a_k)^{-1} \sum_{k=-\infty}^{n_j} a_k \leq \frac{2^{n_j} \varphi(2^{n_j}) - \Phi(2^{2n_j})}{\Phi(2^{n_j})} \leq \frac{t_j \varphi(t_j) - \Phi(t_j)}{\Phi\left(\frac{t_j}{2}\right)} \leq \frac{1}{2j}.
\]

Then for \(k_0 = n_j - j + 1\), we have

\[
(\sum_{k=k_0+1}^{n_j} 2^{n_j-k+1}a_k)^{-1} \sum_{k=-\infty}^{n_j} a_k \geq \frac{1}{2^{j-1}}, \quad (\sum_{k=-\infty}^{k_0} 2^{n_j-k+1}a_k)^{-1} \sum_{k=-\infty}^{n_j} a_k \leq \frac{1}{2^{j-1}}.
\]

Thus \(\sum_{k=-\infty}^{n_j} 2^{-k}a_k \leq 2 \sum_{k=-\infty}^{k_0} 2^{-k}a_k\) and

\[
\frac{\varphi(\frac{t_j}{2})}{\int_0^{2t_j} \frac{\varphi(s)}{s} ds} \leq \left(\frac{1}{2}\right) \sum_{k=-\infty}^{n_j} (n_j-k+1)2^{-k}a_k \leq \frac{8}{(n_j-k_0+1)} = \frac{8}{j}.
\]

Therefore

\[
j \varphi\left(\frac{t_j}{2}\right) \leq 8 \int_0^{2t_j} \frac{\varphi(s)}{s} ds \leq 8c_1 \varphi(2c_1 t_j)
\]

and

\[
j \Phi\left(\frac{t_j}{2}\right) \leq j \varphi\left(\frac{t_j}{2}\right) \leq 4c_1 t_j \varphi(2c_1 t_j) \leq \Phi(6c_1 t_j), \forall j > 0,
\]

a contradiction that proves the necessity.

**Theorem 4.** Let \(\Phi\) be a nondecreasing convex function on \([0, \infty)\) with \(\Phi(0) = 0\) and \(\Phi \in \Delta_2\). Then \(\lim_{t \to 0^+} t^p \Phi(t) > 1\) if and only if there exist some \(s_0, c_1 > 0\) such that (8) holds.

**Proof.** Without loss of generality, we can assume that there exists \(t_0 > 0\) such that \(t_0 \psi(t) > \Psi(t)\), \(\Psi(2t) \leq c \Psi(t), \forall t > t_0\). Now, for any \(s_0 > t_0\), consider the function \(\tilde{\Psi}(t) = \Psi(t)\chi_{[s_0, \infty)} + \Psi(s_0)(\frac{t}{s_0})^\alpha \chi_{(0, s_0]}\), where \(\alpha = \frac{s_0 \psi(s_0)}{\Psi(s_0)} > 1\). Applying Lemma 3 to \(\tilde{\Psi}(t)\) we get

\[
\inf_{t > s_0} \frac{t \psi(t)}{\Psi(t)} > 1 \text{ if and only if } \exists c > 0 \text{ such that } \int_{s_0}^{t} \frac{\psi(s)}{s} ds < c \psi(ct), \forall t > s_0.
\]

The proof is finished.

The following are some examples which make the inequalities in this paper hold:

**Example 1.** For \(1 < p < \infty\), let

\[
\Phi(t) = \Psi(t) = \begin{cases} t, & t < 1, \\ t^p, & t \geq 1. \end{cases}
\]
In this case,
\[ \varphi(t) = \psi(t) = \begin{cases} 
1, & t < 1, \\
p t^{p-1}, & t \geq 1.
\end{cases} \]

Example 2. Let
\[ \Phi(t) = t, \Psi(t) = t \log^+ t. \]
In this case
\[ \varphi(t) = 1, \psi(t) = \begin{cases} 
0, & t < 1, \\
1 + \log^+ t, & t \geq 1.
\end{cases} \]

Example 3. Suppose that \( n \geq 1 \) and
\[ \Phi(t) = t (\log^+ t)^+, \quad \Psi(t) = t (\log^{n+1} t)^+. \]
In this case
\[ \varphi(t) = \begin{cases} 
0, & 0 \leq t < 1, \\
n \log^{n-1} t + \log^+ t, & t \geq 1,
\end{cases} \]
\[ \psi(t) = \begin{cases} 
0, & 0 \leq t < 1, \\
(n+1) \log^+ t + \log^{n+1} t, & t \geq 1.
\end{cases} \]

References