1. (42 pts) Fill the blanket (2pts each blanket for (a)-(j), 3pts each blanket for (k-l), No procedure needed).

(a) \( S = \{ \text{all rational numbers smaller than } \pi \} \), sup \( S = \) ............

(b) \( S = \{ 2 - \frac{(-1)^n}{n}, n \in \mathbb{N} \} \), sup \( S = \) ............., inf \( S = \) ............. .

(c) Let \( V_{\frac{1}{2}}(4) \) be the \( \frac{1}{2} \)-neighborhood of 4, \( V_1(3) \) be 1-neighborhood of 3, sup \( V_{\frac{1}{2}}(4) \cup V_1(3) = \) ............., inf \( V_1(3) \cap V_{\frac{1}{2}}(4) = \) .............

(d) \( \lim(3 + \frac{2}{n})^2 = \) ............

(e) \( \lim \frac{n^2}{2^n} = \) .............

(f) \( \lim_{x \to 0} \frac{x^2}{|x|} = \) .............

(g) \( \lim_{x \to -1} (x + 1) \sin\left(\frac{1}{x+1}\right) = \) .............

(h) \( f(x) = \frac{2}{x} - x, x \neq 0 \), the derivative \( f'(x) = \) .............

(i) \( f(x) = \sin 3x \), the derivative \( f'(x) = \) .............

(j) \( \int_0^1 (\sin t^2)2tdt = \) .............

(k) \( \int_0^\pi (\sin t)tdt = \) .............
(1) Consider the following functions defined on $(-1, 1)$, $f(x) = |x|$, $g(x) = x^2$, $h(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $u(x) = \frac{1}{x}$ for $x \neq 0$, $v(x) = x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $w(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $h(0) = u(0) = v(0) = w(0) = 0$.

Which of these functions are bounded on $(-1, 1)$?

Which of these functions are integrable on $(-1, 1)$?

Which of these functions are continuous on $(-1, 1)$?

Which of these functions are uniformly continuous on $(-1, 1)$?

Which of these functions are differentiable on $(-1, 1)$?

2. (5pts) Prove $\mathbb{Z}$ the set of all integer is a countable set.
3. (9pts) Let $x_1 = \frac{5}{3}$ and $x_{n+1} = 3 - \frac{2}{x_n}$ for all $n \in \mathbb{N}$.
   (i) (4pts) Prove by induction that $2 \leq x_n \leq 3$ for all $n \in \mathbb{N}$.

(ii) (5pts) Prove that $(x_n)$ is convergent and find the limit of $x_n$. (hint: use monotone convergence theorem or show $(x_n)$ is a contractive sequence)
4. (8pts) Consider \( f = \sin(1/x) \) for \( x \in (0, 1) \). Prove that \( \lim_{x \to 0} f(x) \) does not exist.

5. (8pts) Consider function \( f = 1 \) for \( x \in [0, 1] \), \( f = 5 \) for \( x \in (1, 2] \). Prove by definition that \( f \) is Riemann integrable on \([0, 2]\).
6. (8pts) Consider function $f$ defined by $f = 2x$ for $x \in [0, 1]$ rational and $f(x) = 0$ for $x$ irrational. Prove $f$ is not Riemann integrable on $[0, 1]$ by Cauchy Criterion.

7. (20 pts) Consider $f(x) = 2x^2$, $x \in \mathbb{R}$
   
   (i)(5pts) Use definition to prove that $f$ is continuous on $\mathbb{R}$. 

(ii) (3pts) Use definition to find the derivative of $f(x)$.

(iii) (2pts) Find $\int_0^1 f$.

(iv) (10pts) Suppose $g$ is another function continuous on $[0, 1]$ and $\int_0^1 g = \int_0^1 f$. Prove that there exist a $x_0 \in (0, 1)$ such that $g(x_0) = 2x_0^2$. 
**Bonus question (10pts)** The so-called Thomaes function $f$ on $[0,1]$ is defined as follows. Let $f(0) = 1$ (for convenience). Let $f(x) = 0$ if $x$ is irrational, $f(x) = \frac{1}{n}$ if $x \neq 0$ is rational and $x = \frac{m}{n}$ for some integer $m,n$ with no common factor except $1$. Prove Thomaes function $f$ is Riemann integrable on $[0,1]$ by showing that the dis-continuous point set of $f$

$$E = \{x \in [0,1], f \text{ is not continuous at } x\}$$

is a null set.