1. (10 pts) Let $S \subset \mathbb{R}$ be a bounded set. Denote $s^* = \sup S$. Prove that $\sup(S \cup \{u\}) = \max\{s^*, u\}$ for any $u \in \mathbb{R}$.

Proof 1: Case (i), $u \geq s^*$. In this case, $\max\{s^*, u\} = u$. Note $u \geq s^* = \sup S \geq s$ for any $s \in S$ and $u \geq u$. We get $u \geq x$ for any $x \in S \cup \{u\}$. So $u$ is an upper bound of $S \cup \{u\}$. On the other hand, for every $\epsilon > 0$, $u - \epsilon < u$ and $u \in S \cup \{u\}$. By Lemma 2.3.4, $u = \sup S \cup \{u\}$.

Case (ii), $u < s^*$. In this case, $\max\{s^*, u\} = s^*$. Since $s^* = \sup S \geq s$ for every $s \in S$ and $s^* \geq u$ in this case, we get $s^* \geq x$ for every $x \in S \cup \{u\}$. So $s^*$ is an upper bound of $S \cup \{u\}$. On the other hand, because $s^* = \sup S$, by Lemma 2.3.4, we have that, for every $\epsilon > 0$, there exists a $s_\epsilon \in S$ such that $s^* - \epsilon < s_\epsilon$. Since $s_\epsilon \in S \cup \{u\}$ also, by Lemma 2.3.4 again, we get $s^* = \sup(S \cup \{u\})$. $\blacksquare$

Proof 2: Since $\max\{s^*, u\} \geq u$ and $\max\{s^*, u\} \geq s^* \geq s$ for any $s \in S$, we get $\max\{s^*, u\}$ is an upper bound of $S \cup \{u\}$. So

$$\max\{s^*, u\} \geq \sup(S \cup \{u\}).$$

(\*)

Conversely, by definition of sup, $\sup(S \cup \{u\}) \geq u$ since $u \in S \cup \{u\}$. And $\sup(S \cup \{u\}) \geq s$ for any $s \in S$. So $\sup(S \cup \{u\})$ is an upper bound of $S$ which implies $\sup(S \cup \{u\}) \geq \sup S = s^*$. Therefore

$$\sup(S \cup \{u\}) \geq \max\{s^*, u\}.$$ 

(**)

Combine (\*) and (**), we get $\sup(S \cup \{u\}) = \max\{s^*, u\}$. 
2. (10 pts) For any \( n \in \mathbb{N} \), let \( I_n = V_\frac{1}{n} (\frac{1}{n}) \) that is the \( \frac{1}{n} \)-neighborhood of \( \frac{1}{n} \). Prove that \( \bigcap_{n=1}^{\infty} I_n = \emptyset \).

**Proof 1:** By definition of \( \epsilon \)-neighborhood, \( V_\frac{1}{n} (\frac{1}{n}) = \{ x, |x - \frac{1}{n}| < \frac{1}{n} \} = (0, \frac{2}{n}) \). Suppose there exists a real number \( x \in \bigcap_{n=1}^{\infty} I_n \) then \( x > 0 \) and

\[
x < \frac{2}{n}
\]

for any \( n \in \mathbb{N} \). Multiple by the positive number \( \frac{n}{x} \) in both sides of the inequality above, we get

\[
n < \frac{2}{x}
\]

for any \( n \in \mathbb{N} \). This contradicts with Archimedean Property 2.4.3. Therefore, there does not exist any real number in \( \bigcap_{n=1}^{\infty} I_n \). That means \( \bigcap_{n=1}^{\infty} I_n = \emptyset \). ■

**Proof 2:** By definition of \( \epsilon \)-neighborhood, \( V_\frac{1}{n} (\frac{1}{n}) = \{ x, |x - \frac{1}{n}| < \frac{1}{n} \} = (0, \frac{2}{n}) \). Suppose there exists a real number \( x \in \bigcap_{n=1}^{\infty} I_n \) then \( x > 0 \) and

\[
x < \frac{1}{n}
\]

for any \( n \in \mathbb{N} \). By Theorem 3.2.5, we get that \( x \leq \lim \frac{1}{n} = 0 \) which contradicts with \( x > 0 \). Therefore, \( \bigcap_{n=1}^{\infty} I_n = \emptyset \). ■