Name ..............................................

1. (10 pts) Let \( f : (-1, 1) \to \mathbb{R} \) be a function continuous at \( c = 0 \). Suppose \( f(0) < 0 \). Prove that there exists a \( \delta > 0 \) such that \( f(x) < 0 \) for any \( x \in (-1, 1) \) with \( |x| < \delta \).

Solution: Since \( f \) is continuous at 0, for every \( \epsilon \), there exists \( \delta > 0 \) such that

\[
|f(x) - f(0)| < \epsilon
\]

for every \( x, x \in (-1, 1), |x| < \delta \). In particular, this is true for \( \epsilon = -\frac{f(0)}{2} \), we then get a \( \delta \) such that, for every \( x, x \in (-1, 1), |x| < \delta \),

\[
|f(x) - f(0)| < -\frac{f(0)}{2}.
\]

Then

\[
\frac{f(0)}{2} < f(x) - f(0) < -\frac{f(0)}{2}.
\]

Therefore

\[
f(x) < \frac{f(0)}{2} < 0.
\]

\[\blacksquare\]

2. (10 pts) Give a brief explaining that \( f(x) = x^2 - 3x + 1 \) is continuous on \( \mathbb{R} \) by using the fact \( g(x) = x \) and \( h(x) \equiv 1 \) are continuous on \( \mathbb{R} \). Prove that \( f(x) = 0 \) has at least 2 roots on \( (0, 3) \).

Solution: Since \( g \) is continuous we know that \( g^2 = g \cdot g \) is continuous. Then \( f = g^2 - 3g + h \) is continuous by Theorem 5.2.1.
Note $f(0) = 1 > 0, f(1) = -1 < 0, f(3) = 1 > 0$. By Theorem 5.3.5 we know there exist $c_1 \in (0, 1), c_2 \in (1, 3)$ such that $f(c_1) = 0, f(c_2) = 0$. Therefore, $f(x) = 0$ has at least 2 roots on $(0, 3)$. ☐