We have \[0 \leq \int \| (f + tg)^2 \| = \| f \|^2 + t^2 \| g \|^2 + 2t \langle f, g \rangle\]

Since \(at^2 + 2bt + c > c - b^2/a\) we get
\[0 \leq -\frac{\langle f, g \rangle^2}{\| g \|^2} + \| f \|^2\]
or
\[\langle f, g \rangle^2 \leq \| f \|^2 \| g \|^2\]

By Schwarz inequality we get
\[f(l) - f(0) = \int_0^l f'(x) \cdot 1 \, dx = \langle 1, f' \rangle \leq \| f \| \| 1 \| = l \| f \|\]

Solution of diffusion equation is given by the series
\[u(x, t) = \sum A_n \sin \frac{n\pi x}{l} e^{-n^2 \pi^2 kt/l}\]
where \(A_n\) are Fourier coefficients on the continuous initial data so they are uniformly bounded by some constant. To assure differentiability of the solution we need to provide the uniform convergence of the series
\[\frac{\pi x}{l} \sum n A_n \cos \frac{n\pi x}{l} e^{-n^2 \pi^2 kt/l}\]

The latter is true since \(ne^{-an^2} = e^{-an^2 + an} < e^{-a(n-1)^2} < e^{-an}\) for large enough \(n\).
To prove the statement we need the following identity:

\[
\frac{1}{\sqrt{4\pi\sigma}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4\sigma} \cos \omega y dy = e^{-\omega^2 \sigma} \cos \omega x
\]  

(1)

This can be done by the following calculation: \( \cos \omega y = \frac{1}{2} e^{i\omega y} + e^{-i\omega y} \) thus

\[
e^{-(x-y)^2/4\sigma} e^{i\omega y} = e^{-(x^2+y^2-2xy-4i\sigma y)/4\sigma} = e^{-(y^2-2y(x+2i\sigma)+x(2i\sigma)^2-4i\sigma)/4\sigma} =
\]

\[
e^{-((x+2i\sigma)-y)^2/4\sigma+\sigma y^2-ix\omega} = e^{-(x+2i\sigma)-y)^2/4\sigma e^{-\sigma y^2} e^{ix\omega}
\]

Second multiplier does not depend on \( y \) so it could be taken out of the integral. Integration of the remained term gives a constant.

Second term goes by the same way

\[
e^{-(x-y)^2/4\sigma} e^{-i\omega y} = e^{-(x-2i\sigma)-y)^2/4\sigma e^{-\sigma y^2} e^{-ix\omega}
\]

Summation gives the desired result (1).

Now starting with the 2l periodic extension of the initial data \( \phi(x) \) we get the solution of the diffusion equation on the whole line given by the formula:

\[
u(x,t) = \frac{1}{4\pi kt} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4\sigma} \phi(y) dy = \sum_n A_n \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4\sigma} \cos \frac{n\pi y}{l} dy
\]

where \( A_n \) are the Fourier coefficients of \( \phi(y) \).

Using (1) with \( \sigma = kt \) and \( \omega = n\pi/l \) we derive

\[
u(x,t) = \sum_n A_n e^{-n^2 \pi^2 l} \cos \frac{n\pi x}{l}
\]

QED

Page 145. ex. 15.

Hint (a): \( \sin \frac{n\pi(x+ct)}{l} + \sin \frac{n\pi(x-ct)}{l} = 2 \sin \left( \frac{n\pi(x+ct)}{l} + \frac{n\pi(x-ct)}{l} \right) \cos \left( \frac{n\pi(x+ct)}{l} - \frac{n\pi(x-ct)}{l} \right) \)

Hint (b): \( \int_{x-ct}^{x+ct} \sin \frac{n\pi s}{l} ds = \cos \frac{n\pi(x-ct)}{l} - \cos \frac{n\pi(x+ct)}{l} \)
Page 150. ex. 1.

Use method of shifting the data: $v(x, t) = u(x, t) - 1$. We get

$$v_t = v_{xx}, \quad v(x, 0) = x^2 - 1$$

with boundary conditions $v_x(0, t) = 0$, $v(1, t) = 0$.

Eigenfunctions has the form $f_n = \cos(\pi x(n + 1/2))$. and so

$$u(x, t) = v(x, t) + 1 = 1 + \sum A_n \cos(\pi x(n + 1/2))e^{-(n+1/2)^2\pi^2 t}$$

where

$$A_n = \int_0^1 (x^2 - 1) \cos((n + 1/2)\pi x)dx$$

Taking this integral by parts three times we get: $A_n = 4(-1)^n+1(n+1/2)^{-3}\pi^{-3}$.

Page 150. ex. 6.

Represent the solution in form of sine series

$$u(x, t) = \sum u_n(t) \sin \frac{n\pi x}{l}$$

From the equation we get

$$\ddot{u}_n + \frac{c^2 n^2 \pi^2}{l^2} u_n = g_n \sin \omega t$$

where $g_n$ is the Fourier coefficient for the function $g(x)$

Solving the latter second order ODE for $u_n(t)$ we derive that for $\omega = \frac{n\pi c}{l}$ the resonance occurs giving the growing with time coefficients

$$u_n(t) = -\frac{A_n}{2\omega}(t \cos \omega t - 1/\omega \sin \omega t)$$