Page 31. ex. 1.

a) \(a_{11} = 1, a_{22} = 1, a_{12} = -2\) thus \(a_{11}a_{22} - a_{12}^2 = 1 - 4 = -3 < 0\). So the equation is hyperbolic.

b) \(a_{11}a_{22} - a_{12}^2 = 9 - 9 = 0\). Hence equation is parabolic.

Page 31. ex. 3.

Any rotation is given by the change of variables \(\xi = \cos(\alpha)x - \sin(\alpha)y\), \(\eta = \sin(\alpha)x + \cos(\alpha)y\). Thus \(u_x = \cos \alpha u_\xi + \sin \alpha u_\eta\) and \(u_y = -\sin \alpha u_\xi + \cos \alpha u_\eta\). Similarly
\[
\begin{align*}
  u_{xx} &= \cos^2 \alpha u_{\xi\xi} + \sin^2 \alpha u_{\eta\eta} + 2 \sin \alpha \cos \alpha u_{\xi\eta} \\
  u_{yy} &= \sin^2 \alpha u_{\xi\xi} + \cos^2 \alpha u_{\eta\eta} - 2 \sin \alpha \cos \alpha u_{\xi\eta} \\
  u_{xy} &= \cos \alpha \sin \alpha (u_{\eta\eta} - u_{\xi\xi}) + (\cos^2 \alpha - \sin^2 \alpha)u_{\xi\eta}
\end{align*}
\]

Thus for \(b_1u_x + b_2u_y = b_1u_\xi + b_2u_\eta\) to be satisfied for all \(\alpha\) one needs \(b_1 = b_2 = 0\). For \(2a_{12}u_{xy} = 2a_{12}u_{\xi\eta}\) one needs \(a_{12} = 0\) and for \(a_{11}u_{xx} + a_{22}u_{yy} = a_{11}u_{\xi\xi} + a_{22}u_{\eta\eta}\) one needs \(a_{11} = a_{22}\).

Page 31. ex. 4.

Since \(a_{11}a_{22} - a_{12}^2 = 4 - 4 = 0\) the equation is parabolic.

For \(u = f(y + 2x) + xg(y + 2x)\) we get
\[
\begin{align*}
  u_{xx} &= 4f''(y + 2x) + 4g'(y + 2x) + 4xg''(y + 2x) \\
  u_{yy} &= f''(y + 2x) + xg''(y + 2x) \\
  u_{xy} &= 2f''(y + 2x) + g'(y + 2x) + 2xg''(y + 2x)
\end{align*}
\]

Therefore
\[
u_{xx} + 4u_{yy} - 4u_{xy} = (4f'' + 4g' + 4xg') + (4f'' + 4xg'') - (8f'' + 4g' + 8xg'') = 0
\]
Page 31. ex. 5.
This is just a simple calculation!

Page 38. ex. 1.
Using the solution formula we get

\[ u(x, t) = \frac{1}{2} (e^{x+ct} + e^{x-ct}) + \frac{1}{2c} \int_{x-ct}^{x+ct} \sin \xi d\xi = e^x \cosh ct + \frac{1}{c} \sin x \sin ct \]

Page 38. ex. 2.
\[ u(x, t) = \frac{1}{2} \log(1 + (x - ct)^2)(1 + (x + ct)^2) + 4t + tx \]

Page 38. ex. 4.
Change of variables \( \xi = x + ct, \eta = x - ct \) immediately gives

\[ u_{\xi\eta} = 0 \]

which implies the statement.

Page 41. ex. 1.
Energy function \( E(t) \) is given by the formula \( E(t) = \frac{1}{2} \int (\rho u_t^2(x, t) + Tu_x^2(x, t)) dx \).
Thus for \( u(x, 0) \equiv 0 \) and \( u_t(x, 0) \equiv 0 \) we get \( E(0) = 0 \). Thanks to energy conservation we get \( E(t) \equiv 0 \). But since energy is non negative by first vanishing theorem this means that \( \rho u_t(x, t)^2 + Tu_x^2(x, t) \equiv 0 \) which implies that \( u_t = u_x = 0 \). Thus \( u(x, t) = \text{const} \). Applying initial conditions we get \( \text{const} = 0 \).

Page 41. ex. 2.

a) \( e_t = u_{tt} u_t + u_{tx} u_x, p_x = u_{tx} u_x + u_t u_{xx} = u_{tx} u_x + u_t u_{tt} \) since \( c = 1 \).

b) Using a) \( e_t t = (e_t)_t = (p_x)_t = (p_t)_x = e_{xx} \).
Page 41. ex. 3.

a) \( u_x(x-y, t) = u_{(x-y)(x-y)}(x-y, t) = c^2 u_{tt}(x-y, t) \).

b) Let \( v(x, t) = u_x(x, t) \). Then \( v_{xx}(x, t) = u_{xxx}(x, t) = (u_{xx}(x, t))_x = (c^2 u_{tt}(x, t))_x = c^2 u_{xt}(x, t) \).

c) Let \( v(x, t) = u(ax, at) \). Then \( v_{xx}(x, t) = a^2 u_{xx}(x, t) = a^2 c^2 u_{tt}(x, t) = c^2 (v_{tt}(x, t)) \).

Page 41. ex. 4.

Using the solution formula we get

\[
u(x+h, t+k) + u(x-h, t-k) = \frac{1}{2} \left( \phi(x+h-t-k) + \phi(x+h+t+k) + \phi(x-h-t+k) + \phi(x-h+t-k) \right) + \]
\[
+ \frac{1}{2} \int_{x+h-t-k}^{x+h+t+k} \psi(y) \, dy + \int_{x-h-t+k}^{x-h+t+k} \psi(y) \, dy
\]

which is precisely the same formula for the right hand side of the identity.