How many loops are on this surface?

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Problem
A collection of curves $\Gamma$ on a surface $S$ can be described by the number of times it meets each edge of a triangulation $\mathcal{T}$ [4]. If $\mathcal{T}$ has edges $e_1, \ldots, e_k$ let $\varpi = \varpi(\Gamma) := (\iota(\Gamma, e_1) \cdots \iota(\Gamma, e_k))$.

Given $\varpi$ decide how many components $\Gamma$ has.

Examples
In each case: how many curves are there? How did you figure this out?

The Street Algorithm [3]

Key observation: If $|\varpi|$ is very large then there are many arcs in each triangle and therefore there must be many long sections where the curves run parallel.

We keep track of these parallel regions, each of which is either a rectangle or an annulus. We refer to these as open and closed streets respectively. In fact there are only ever at most $2k$ streets.

Start with an open street for each type of arc in each triangle
while there is an open street do
if a street meets itself exactly then
replace it with a closed street
else if a street meets itself then
replace it with a spiral
else
merge all open streets
end if
end while

At the end, each street is an annulus and the number of components of $\Gamma$ is the sum of their widths. However, after the outer loop is repeated a constant number of times the width of the widest open street drops by a definite fraction.

Hence, this only requires $O(\log(|\varpi|))$ time and space.

Uses
We can surger $S$ along $\Gamma$ to obtain a new surface $S_{\mathcal{T}}$. If $\mathcal{T}$ is a triangulation of $S$ then it can also be surgered, giving a triangulation $\mathcal{T}_{\mathcal{T}}$ of $S_{\mathcal{T}}$.

Lemma (Bell). We can compute $\mathcal{T}_{\mathcal{T}}$ from $\mathcal{T}$ in $O(\log(|\varpi(\Gamma)|))$ time and space.

This is a very useful technique as it allows us to construct arguments based on induction on the genus of $S$. For example:

Theorem (Bell [2]). Suppose that $H$ is a finite set of homeomorphisms of $S$. There is a computable constant $K$ such that for each $h \in (H)$ there is a collection of curves $\Gamma$ fixed by $h$ (up to isotopy) such that the homeomorphism induced by $h$ on $S_{\mathcal{T}}$ fixes no curves and

$$\log(|\varpi(\Gamma)|) \leq K \ell(h).$$

Naïve Algorithm

while there is an unmarked arc do
Get an unmarked arc
repeat
Mark the current arc
Follow along to next arc
until back at the starting point
end while

The number of components of $\Gamma$ is the number of times the outer loop is repeated. However, this involves crossing every arc in every triangle.

Hence, this requires $O(|\varpi|)$ time and space.

References