Homework Assignment 2

Due: Wednesday March 13, 2013

For the following, $G$ denotes a locally compact group with fixed left Haar measure $dg$. Recall that $S_{\sigma}(L^{\infty}(G)) = \{ f \in L^{1}(G) : \| f \|_{L^{1}(G)} \leq 1, f \geq 0 \}$.

1. Let $m$ be a left-invariant mean on $L^{\infty}(G)$. Show that $m|_{C_{0}(G)} = 0$ if $G$ is not compact.

2. Prove that $G$ is amenable if and only if there exists a bi-invariant mean on $L^{\infty}(G)$. I.e., $m \in S(L^{\infty}(G))$ such that $m(\varphi) = m(L_{g}\varphi) = m(R_{g}\varphi)$ for each $\varphi \in L^{\infty}(G)$, $g \in G$. (Hint: If $G$ is amenable, there is a net $(f_{\alpha})_{\alpha} \subset S_{\sigma}(L^{\infty}(G))$ such that $\| L_{g}f_{\alpha} - f_{\alpha} \|_{1} \to 0$ uniformly on compact subsets of $G$. Now consider the net $(f_{\alpha} * f^{*}_{\alpha})_{\alpha} \subset S_{\sigma}(L^{\infty}(G)))$.

3. A uniformly bounded representation of $G$ on a Hilbert space $H$ is a SOT-continuous (not necessarily unitary) group homomorphism $\sigma : G \to B(H) \cap GL(H)$ such that $\| \sigma \|_{\infty} = \sup_{g \in G} \| \sigma(g) \| < \infty$.
   (a) Assume $G$ is amenable with left-invariant mean $m \in S(L^{\infty}(G))$ and define
   $$\langle \xi|\eta \rangle_{\sigma} := \int_{G} \langle \sigma(g)^{-1}\xi|\sigma(g)^{-1}\eta \rangle dm(g) \quad (\xi, \eta \in H).$$
   Show that $\langle \cdot | \cdot \rangle_{\sigma}$ defines an inner product on $H$ and that the associated norm $\xi \mapsto \| \xi \|_{\sigma} = \langle \xi|\xi \rangle_{\sigma}^{1/2}$ is equivalent to the original norm on $H$.
   (b) Show that with respect to the new inner product $\langle \cdot | \cdot \rangle_{\sigma}$, $\sigma$ becomes a unitary representation. (That is, uniformly bounded representations of amenable groups are unitarizable.)

4. Let $\mu$ be a positive (possibly infinite) Radon measure on $G$. Assume that $\mu$ has the property that $\mu * L^{2}(G) \subseteq L^{2}(G)$.
   (a) Show that $\mu$ determines an element $\lambda(\mu) \in VN(G) \subset B(L^{2}(G))$ defined by $\lambda(\mu)\xi = \mu * \xi$ ($\xi \in L^{2}(G)$). (You may use the fact (without proof) that $VN(G) = \rho(G)'$).
   (b) Show that if $G$ is amenable, then $\| \lambda(\mu) \| = \mu(G)$ (so $\mu$ is automatically finite). (Hint: Make use of the fact that there exists a net $(\xi_{\alpha})_{\alpha}$ of non-negative $\| \cdot \|_{2}$-unit vectors in $C_{c}(G)$ such that $\varphi_{\alpha} = \xi_{\alpha} * \xi^{*}_{\alpha} \to 1$ uniformly on compact subsets of $G$.)

Remark: If $G$ is not amenable, then there may exist infinite measures $\mu$ for which $\lambda(\mu)$ is bounded. We will see this phenomenon for free groups later. Another case where this happens is the Kunze-Stein Phenomenon: If $G$ is a semisimple Lie group with finite centre and $1 \leq p < 2$, then $L^{p}(G) * L^{2}(G) \subseteq L^{2}(G)$.
5. **Leptin’s Theorem** states that $G$ is amenable if and only if the Fourier algebra $A(G)$ has a bounded approximate identity $(\varphi_\alpha)_\alpha$. We will (partially) prove this theorem.

(a) Suppose $(\varphi_\alpha)_\alpha \subset A(G)$ is a BAI with $\sup_\alpha \|\varphi_\alpha\|_{A(G)} = M < \infty$. Show that $\varphi_\alpha \to 1$ uniformly on compact subsets of $G$.

(b) Using (5a), prove that $\lambda : C^*(G) \to C^*_\lambda(G)$ is injective (and therefore $G$ is amenable) by adapting the argument used in class on February 20.

(c) For the converse, we will assume that $G$ is discrete, so that $C_c(G) = \{f : G \to \mathbb{C} \mid \text{supp } f \text{ is finite}\}$. Show that $C_c(G) \subset A(G)$ and $\|\cdot\|_{A(G)} = A(G)$.

(d) Assume $G$ is now discrete and amenable, and let $(\varphi_\alpha)_\alpha \subset P_1(G) \cap C_c(G)$ be a net such that $\varphi_\alpha \to 1$ uniformly on compacta (= pointwise, because $G$ is discrete). Show that $(\varphi_\alpha)_\alpha$ is a contractive approximate identity for $A(G)$.

**Remark:** If $G$ is not discrete, the net $(\varphi_\alpha)_\alpha$ from (5d) is still a BAI for $A(G)$, but the proof is much more involved.

6. Let $X_i \subset B(H_i)$ be operator spaces and $\varphi_i \in X_i^*$ $(i = 1, 2)$. Prove that

$$\varphi_1 \otimes \varphi_2 : X_1 \otimes X_2 \to \mathbb{C}; \quad (\varphi_1 \otimes \varphi_2)(x_1 \otimes x_2) = \varphi_1(x_1)\varphi_2(x_2) \quad (x_1 \in X_1, \ x_2 \in X_2)$$

extends to a bounded linear functional $\varphi_1 \otimes \varphi_2 : X_1 \otimes_{\min} X_2 := \overline{X_1 \otimes X_2}_{\|\cdot\|_{B(H_1 \otimes H_2)}} \to \mathbb{C}$ with $\|\varphi_1 \otimes \varphi_2\| = \|\varphi_1\|\|\varphi_2\|$. (Hint: Recall that $\varphi_1, \varphi_2$ are completely bounded maps).

7. Let $A$ be a $C^*$-algebra and $\Phi : A \to B(H)$ a contractive completely positive map with Stinespring decomposition $\Phi = V^*\pi(\cdot)V$.

(a) Prove the **Schwarz-inequality**: $\Phi(a)^*\Phi(a) \leq \Phi(a^*a)$ for all $a \in A$.

(b) Prove the **bimodule property**: If $a \in A$ has the property that $\Phi(a)^*\Phi(a) = \Phi(a^*a)$ and $\Phi(aa^*) = \Phi(a)\Phi(a)^*$, then

$$\Phi(ba) = \Phi(b)\Phi(a) \quad \& \quad \Phi(ab) = \Phi(a)\Phi(b) \quad (b \in A).$$

8. Use the structure theorem for completely bounded maps to give a simple proof of the fact that the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$ of a locally compact group is a Banach algebra.