Section 3.4, #14 [3 points] (a) The IVP associated to the mass-spring system is
\[ 25x'' + 10x' + 226x = 0, \quad x(0) = 20, x'(0) = 41. \]
The roots of the characteristic polynomial \( P(r) = 25r^2 + 10r + 226 \) are \(-\frac{1}{5} \pm 3i\), so the solution is
\[ x(t) = c_1 e^{-\frac{1}{5}t} \cos(3t) + c_2 e^{-\frac{1}{5}t} \cos(3t) = Ce^{-\frac{1}{5}t} \cos(3t - \alpha). \]
Where \( C = \sqrt{c_1^2 + c_2^2}, \sin \alpha = \frac{c_1}{C}, \cos \alpha = \frac{c_2}{C} \). Plugging in the IC’s we can solve for \( c_1, c_2 \) (or \( C, \alpha \)), and we get
\[ x(t) = 25e^{-\frac{1}{5}t} \cos(3t - \alpha), \quad \alpha = \arctan(3/4). \]
From this equation, the solution does indeed have the graph given by Figure 3.4.15. Note that this system is underdamped, since it has complex roots (and therefore oscillates about its steady state equilibrium).

(b). The pseudoperiod of the oscillation of \( x(t) \) is just the period of the \( \cos(3t - \alpha) \) term. This is the period that the solution would have if it did not have the decaying amplitude factor of \( 25e^{-\frac{1}{5}t} \). The pseudoperiod is therefore
\[ T = \frac{2\pi}{\omega} = \frac{2\pi}{3}. \]
The envelope curves for \( x(t) \) are given by the amplitude factor on \( x(t) \) : they are \( \pm 25e^{-\frac{1}{5}t} \).

Section 3.6, #18 [2 points] We have
\[ mx'' + cx' + kx = F_0 \cos(\omega t), \quad (m = 1, c = 10, k = 650, F_0 = 100). \]
If we apply the method of undetermined coefficients, the particular (steady state solution) of this of the form by
\[ x_p(t) = C(\omega) \cos(\omega t - \alpha) = A \cos(\omega t) + B \sin(\omega t), \]
where the amplitude factor is (refering to equation (21) in the text):
\[ C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{100}{\sqrt{(650 - \omega^2)^2 + 100\omega^2}} \]
To investigate practical resonance, we need to try and maximize the amplitude $C(\omega)$ of the steady state solution. This maximum will occur when $C'(\omega) = 0$. Solving the equation

$$0 = C'(\omega) = -50\frac{(-4\omega)(650 - \omega^2) + 200\omega}{((650 - \omega^2)^2 + 100\omega^2)^{3/2}},$$

we see that $\omega_* = 0, 10\sqrt{6}$ are the only solutions. After plugging in these two values, we see $C(\omega)$ must be maximized at $\omega_* = 10\sqrt{6}$, and this is the practical resonance frequency.

**Section 3.6, #24 [2 points]** If an undamped mass-spring system is acted on by a force $F(t) = F_0\cos(3\omega t)$, we will have resonance whenever $F(t)$ contains a term of the form $A\cos(\omega_0 t)$ or $B\sin(\omega_0 t)$ where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the unforced system. Now, using the product formula $\cos a\cos b = \frac{1}{2} \left( \cos(a+b) + \cos(a-b) \right)$ twice:

$$\cos^3(\omega t) = \cos(\omega t)\frac{\cos(2\omega t)+1}{2} = \cos(3\omega t) + \cos(\omega t)) + \frac{1}{2} \cos(3\omega t) + \frac{3}{4} \cos(\omega t),$$

we see that resonance occurs when $\omega = \omega_0 \text{ OR when } 3\omega = \omega_0$.

**Section 3.7, #24 [1 point]** We have to prove that if $R, L, C$ are all positive, then every solution $I$ of

$$LI'' + RI' + I/C = 0$$

is transient. If we let $m = L, c = R$ and $k = \frac{1}{C}$, then the above equation is precisely the equation of a free damped mass spring system with mass $m$, dashpot constant $c$, and spring constant $k$. However, in class, we already showed that all possible solutions for $m, k, c > 0$ are transient (i.e., tend to 0 as $t \to \infty$). Thus we are done.

**Note:** One can also prove this directly by considering the roots of the characteristic polynomial of the above equation and showing that they always have negative real parts.

**Section 4.1, #8 [2 points]** Let

$$x_1 = x, \quad x_2 = x', \quad y_1 = y, \quad y_2 = y'.$$

Then the given pair of second order equations becomes

$$x'_1 = x_2$$
$$x'_2 = -3x_2 - 4x_1 + 2y_1$$
$$y'_1 = y_2$$
$$y'_2 = -2y_2 + 3x_1 - y_1 + \cos t.$$