INSTRUCTIONS

• This exam 50 minutes long. No personal aids or calculators are permitted.

• Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.

• EXPLAIN YOUR WORK! Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.

• GOOD LUCK!

PLEASE NOTE: “Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.”
1. Consider the ordinary differential equation

\[ Ly = y^{(3)} - 2y'' + 2y' = x + xe^x. \]

(a) (5 points) Determine the complementary function \( y_c(x) \) for the above ODE.

(b) (8 points) Find a particular solution \( y_p(x) \) to the above ODE.
(c) (2 points) Write down the general solution to the above ODE.
2. Consider the ordinary differential equation

\[ Ly = y'' - \frac{1}{x} y' + \frac{1}{x^2} y = x^3. \]

(a) (4 points) Verify that \( y_1(x) = x \) and \( y_2(x) = x \ln |x| \) are solutions to the associated homogeneous ODE

\[ Ly = 0 \quad (x \neq 0). \]

(b) (5 points) Compute the Wronskian \( W(y_1, y_2) \) for the pair of functions \( y_1, y_2 \) above. Are \( y_1 \) and \( y_2 \) linearly independent on the interval \( I = (0, \infty) \)? Why or why not?

(c) (6 points) Find a particular solution to

\[ y'' - \frac{1}{x} y' + \frac{1}{x^2} y = x^3 \quad (x > 0). \]

(Hint: One possible approach is to use the variation of parameters method.)
3. (12 points) For the following endpoint problem, determine all eigenvalues $\lambda \in \mathbb{R}$ and their associated eigenfunctions:

$$y'' - 4y' + \lambda y = 0, \quad y(0) = y(1) = 0.$$
4. Consider a mass-spring-dashpot system with mass $m = 1\text{kg}$, spring constant $k = 4 \text{ N/m}$ and dashpot damping constant $\beta > 0 \text{ N s/m}$. Let $x(t)$ denote the displacement (in metres, at time $t$) of the mass from its equilibrium resting position.

(a) (4 points) For what values of $\beta$ is the system underdamped?

For the remainder of the problem, assume that the dashpot is disconnected from the system (i.e., set $\beta = 0$) and that an external force $F(t) = 2\sin \omega t$ Newtons is applied to the mass.

(b) (2 points) At what forcing frequency $\omega$ will resonance occur in the forced system?

(c) (6 points) Write down the general solution $x(t)$ in this case.