Problem 1. (5 points) Consider a homogeneous linear differential equation with constant real coefficients which has order 6. Suppose \( y(x) = x^2e^{2x}\cos(x) \) is a solution. Write down the general solution.

Solution. The general solution is \( (c_1 + c_2x + c_3x^2)e^{2x}\cos(x) + (c_4 + c_5x + c_6x^2)e^{2x}\sin(x) \).

Problem 2. (20 points) Find the general solution of \( x' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x \).

Solution. The characteristic polynomial \( (1 - \lambda)^2 - 4 \) has roots \( \lambda = -1, 3 \).

For \( \lambda = 3 \), solving \( \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), we find the eigenvector \( v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

For \( \lambda = -1 \), solving \( \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), we find the eigenvector \( v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

Hence, the general solution is \( x(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} \).
Problem 3. (10 points) The position $x(t)$ of a certain mass on a spring is described by $x'' + cx' + 5x = F \sin(\omega t)$.

(a) Assume first that there is no external force, i.e. $F = 0$. For which values of $c$ is the system overdamped?

(b) Now, $F \neq 0$ and the system is undamped, i.e. $c = 0$. For which values of $\omega$, if any, does resonance occur?

Solution.

(a) The discriminant of the characteristic equation is $c^2 - 20$. Hence the system is overdamped if $c^2 - 20 > 0$, that is $c > \sqrt{20} = 2\sqrt{5}$.

(b) The natural frequency is $\sqrt{5}$. Resonance therefore occurs if $\omega = \sqrt{5}$. □

Problem 4. (20 points) Find the general solution of the differential equation $y^{(3)} - y = e^x + 7$.

Solution. Let us first solve the homogeneous equation $y''' - y = 0$. Its characteristic polynomial $r^3 - 1 = (r - 1)(r^2 + r + 1)$ has roots $r = 1$ and $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

There is a particular solution of the form $y_p = Axe^x + B$.

$$
\begin{align*}
y_p' &= (x+1)e^x, \\
y_p'' &= A(x+2)e^x, \\
y_p''' &= A(x+3)e^x
\end{align*}
$$

Plugging into the DE, we get $y'''_p - y_p = 3Ae^x - B = e^x + 7$. Consequently, $A = \frac{1}{3}$, $B = -7$.

Hence, the general solution is $-7 + (c_1 + \frac{1}{3}x)e^x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-x/2} \sin\left(\frac{\sqrt{3}}{2}x\right)$. □
Problem 5. (20 points) Consider, for \( x > 0 \), the second-order differential equation

\[
y'' - \left(1 + \frac{2}{x}\right) y' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y = 0.
\]

(a) Show that the functions \( y_1(x) = x \) and \( y_2(x) = x e^x \) are solutions to this differential equation.

(b) Using the Wronskian, show that \( y_1 \) and \( y_2 \) are linearly independent solutions to the above differential equation.

(c) Find, for \( x > 0 \), the general solution to the second-order differential equation

\[
y'' - \left(1 + \frac{2}{x}\right) y' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y = 2x.
\]

Solution.

(a) We have

\[
y_1'' - \left(1 + \frac{2}{x}\right) y_1' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y_1 = 0 - \left(1 + \frac{2}{x}\right) + \left(\frac{1}{x} + \frac{2}{x^2}\right) x = 0,
\]

and

\[
y_2'' - \left(1 + \frac{2}{x}\right) y_2' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y_2 = x e^x + 2e^x - \left(1 + \frac{2}{x}\right) (x e^x + e^x) + \left(\frac{1}{x} + \frac{2}{x^2}\right) (x e^x) = 0.
\]

(b) The Wronskian of \( y_1 \) and \( y_2 \) is given by

\[
W(x) = \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = \det \begin{bmatrix} x & x e^x \\ 1 & x e^x + e^x \end{bmatrix} = x^2 e^x.
\]

Since \( W(x) \neq 0 \) on the domain of definition for the differential equation, the Wronskian theorem implies that \( y_1 \) and \( y_2 \) are linearly independent.

(c) The general solution is given by

\[
y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x) = c_1 x + c_2 x e^x + y_p(x),
\]

where \( y_p \) is any particular solution to the above non-homogeneous equation. To find such a \( y_p \), we will use the variation of parameters formula:

\[
y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) = u_1(x) x + u_2(x) x e^x,
\]

where

\[
u_1(x) = -\int \frac{2x y_2(x)}{W(x)} \, dx = -\int 2\, dx = -2x,
\]

\[
u_2(x) = \int \frac{2x y_1(x)}{W(x)} \, dx = \int 2 e^{-x} \, dx = -2 e^{-x}.
\]

So,

\[
y(x) = c_1 x + c_2 x e^x - 2x^2 - 2x = d_1 x + d_2 x e^x - 2x^2.
\]
Problem 6. (20 points) The motion of a certain mass on a spring is described by \( x'' + 2x' + 2x = 5 \sin(t) \).

(a) What is the amplitude of the resulting steady periodic oscillations?

(b) Assume that the mass is initially at rest (i.e. \( x(0) = 0 \), \( x'(0) = 0 \)) and find the position function \( x(t) \).

Solution.

(a) The characteristic polynomial of the associated homogeneous DE has roots \( -2 \pm \sqrt{4 - 8} = -1 \pm i \).

Hence, \( x_{sp} \) is the form \( x_{sp} = A_1 \cos(t) + A_2 \sin(t) \).

We compute \( x_{sp}' = -A_1 \sin(t) + A_2 \cos(t) \) and \( x_{sp}'' = -A_1 \cos(t) - A_2 \sin(t) \).

Plugging into the DE gives \( x_{sp}'' + 2x_{sp}' + 2x_{sp} = (A_1 + 2A_2) \cos(t) + (A_2 - 2A_1) \sin(t) = 5 \sin(t) \). Consequently, \( A_1 + 2A_2 = 0 \) and \( A_2 - 2A_1 = 5 \), resulting in \( A_1 = -2 \), \( A_2 = 1 \).

Thus, \( x_{sp} = -2 \cos(t) + \sin(t) \). The amplitude is \( \sqrt{(-2)^2 + 1^2} = \sqrt{5} \).

(b) From first part, we know that \( x(t) = -2 \cos(t) + \sin(t) + e^{-t}(c_1 \cos(t) + c_2 \sin(t)) \).

Using \( x(0) = -2 + c_1 = 0 \) we find \( c_1 = 2 \).

\( x'(t) = 2 \sin(t) + \cos(t) - e^{-t}(2 \cos(t) + c_2 \sin(t)) + e^{-t}(-2 \sin(t) + c_2 \cos(t)) \). Hence, \( x'(0) = 1 - 2 + c_2 = 0 \) results in \( c_2 = 1 \).

In conclusion, \( x(t) = -2 \cos(t) + \sin(t) + e^{-t}(2 \cos(t) + \sin(t)) \). \( \square \)

Problem 7. (5 points) Let \( y_p \) be any solution to the inhomogeneous linear differential equation \( y'' + xy = e^x \). Find a homogeneous linear differential equation which \( y_p \) solves.

Hint: Do not attempt to solve the DE.

Solution. Apply \( \frac{d}{dx} \) to both sides of the differential equation to get \( y''' + xy' + y = e^x \). Subtracting the two differential equations, we get the homogeneous linear DE \( y''' - y'' + xy' + (1 - x)y = 0 \). \( \square \)