INSTRUCTIONS

• This exam is three (3) hours long. No personal aids or calculators are permitted.

• Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.

• EXPLAIN YOUR WORK! Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.

• GOOD LUCK!

PLEASE NOTE: “Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.”

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SOME USEFUL FORMULAS:

\[ e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \ldots \]

\[ x(t) = \Phi(t)\Phi(a)^{-1}x(a) + \Phi(t) \int_a^t \Phi(s)^{-1}f(s)ds \]

\[ \cos(a) \cos(b) = \frac{1}{2}(\cos(a + b) + \cos(a - b)) \]

\[ \sin(a) \sin(b) = \frac{1}{2}(\cos(a - b) - \cos(a + b)) \]

\[ \sin(a) \cos(b) = \frac{1}{2}(\sin(a + b) + \sin(a - b)). \]
1. (8 points) Find an explicit solution to the initial value problem

\[
\frac{dy}{dx} = xe^x y^2 + xe^x; \quad y(0) = 0.
\]
2. (a) (3 points) Prove that the differential equation

\[ 1 + y^3 + y \cos(xy) + (3y^2x + x \cos(xy)) \frac{dy}{dx} = 0, \]

is exact.

(b) (5 points) Find an implicit solution to the ODE in part (a).
3. The town of Abnormal, Illinois is infected with a population of zombies. Suppose that the zombie population $P(t)$ grows by infecting healthy humans, while at the same time is partially “harvested” (i.e., destroyed) by the local health authority. Assume that $P(t)$ is modeled by the “harvested logistic equation”

$$\frac{dP}{dt} = (15 - P)(P - 5) \quad \text{(thousands of zombies per day)}.$$

(a) (6 points) Sketch the slope field for this differential equation.

(b) (2 points) If at day 0, the zombie population is 10 thousand, what will the long-term population $P_\infty = \lim_{t \to \infty} P(t)$ of zombies be?

(c) (2 points) For what range of initial populations $P(0)$ will the zombie population $P(t)$ be eventually reduced to zero by the health authority?
4. (a) (6 points) Find the general solution to the differential equation

\[ y''' + y'' + 3y' - 5y = 0. \]

\textbf{(Hint: } r_0 = 1 \text{ is one of the roots of the characteristic polynomial.)}

(b) (5 points) Find a particular solution to the differential equation

\[ y''' + y'' + 3y' - 5y = e^x + 1. \]

(c) (1 point) Write down the general solution to the differential equation in part (b).
5. Let $P(t) = [p_{ij}(t)]$ be an $n \times n$ matrix of continuous functions (on $\mathbb{R}$) and consider the homogeneous first order linear system

$$x' = P(t)x \quad (x(t) \in \mathbb{R}^n).$$

(a) (3 points) Explain what a fundamental matrix is for this linear system.

(b) (3 points) Prove that $\Phi(t) := \exp \left( \int_0^t P(s)ds \right)$ is a fundamental matrix for this linear system. (Here $\exp(B)$ denotes the matrix exponential of a matrix $B$.)

(c) (6 points) Solve the initial value problem

$$x' = \begin{bmatrix} 0 & t^2 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} t \\ 0 \end{bmatrix}; \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
6. (10 points) Find the general solution to the linear system

\[ x' = \begin{bmatrix} 1 & -4 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix} x. \]
7. A $2\pi$-periodic external force of $f(t)$ Newtons is applied to an undamped mass-spring system with mass $m = 3$ kg and spring constant $k = 27$ N/m.

(a) (2 points) Write down the equation of motion for the system and determine the natural frequency of the system.

(b) (5 points) Explain why the numerical quantities

$$\int_{-\pi}^{\pi} f(t) \sin(3t) dt \quad \text{and} \quad \int_{-\pi}^{\pi} f(t) \cos(3t) dt,$$

are relevant to the investigation of pure resonance in this system.

(c) (5 points) If $f(t) = |\sin t|$ for $-\pi < t < \pi$, will pure resonance occur?
(d) (6 points) Compute the Fourier series for the function $f(t)$ from part (c) and use this to find a particular solution $x_p(t)$ for the forced system.
8. Let $L > 0$ and let $f(t) = L - t$ for $0 < t < L$.

(a) (5 points) Determine the Fourier sine series for $f$ on the interval $[0, L]$. 
(b) (5 points) Determine the Fourier cosine series for $f$ on the interval $[0, L]$. 
(c) (12 points) Consider an $L \times L$ square metal plate with vertices $(0, 0), (0, L), (L, L)$ and $(L, 0)$. Suppose that the plate is insulated along its top and bottom edges, the temperature at the left edge is held at 0 degrees, and the temperature along the right edge is held at $u(L, y) = L - y$ degrees. If $u(x, y)$ denotes the steady-state temperature distribution in this plate, then the associated boundary-value problem for $u$ is

\[
\begin{align*}
  u_{xx} + u_{yy} &= 0 & (0 < x, y < L) \\
  u_y(x, 0) &= u_y(x, L) = 0 & (0 < x < L) \\
  u(0, y) &= 0 & (0 < y < L) \\
  u(L, y) &= L - y & (0 < y < L).
\end{align*}
\]

Using the method of separation of variables, find $u(x, y)$. 

(Extra work space.)