Review Problems for Exam 1:
1.09, 2.05, VC.01-VC.06

Exam 1: 10/21/2009
\( R.1 \)

Go with the ellipse \( \left( \frac{x-1}{4} \right)^2 + \left( \frac{y+2}{3} \right)^2 = 1 \)

Write down a parametric formula for the ellipse where \( t \) advances from 0 to \( 2 \pi \).

\( R.2 \)

Give the number \( t \) that makes \( X = \{3, -1, 2\} \) and \( Y = \{1, 1, t\} \) perpendicular.

\( R.3a \)

Here are the vectors \( X = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\} \) and \( Y = \{1, 2.5\} \) with their tails at \( \{0, 0\} \), shown in true scale:

Identify \( X \) and \( Y \), and pencil in the component of \( Y \) in the direction of \( X \). Calculate the component of \( Y \) in the direction of \( X \).

\( R.3b \)

If \( X.Y > 0 \), then is the push of \( X \) in the direction of \( Y \) with \( Y \) or against \( Y \)?
If \( X.Y < 0 \), then is the push of \( X \) in the direction of \( Y \) with \( Y \) or against \( Y \)?

\( R.4 \)

Write a parametric formula for the line that passes through the points \( \{3, 1\} \) and \( \{5, 11\} \).
Give a vector parallel to this line.
Give a vector perpendicular to this line.
R.5) Given $X = \{1, 1, 1\}$ and $Y = \{2, 0, 0\}$, calculate $X \cdot Y$ and $X \times Y$.

R.6) Given that $X$ and $Y$ are perpendicular unit vectors in 2D, describe the curve traced out by

$$P[t] = \cos[t] X + \sin[t] Y$$

as $t$ runs from 0 to $2\pi$.

R.7) You are walking around a closed curve with no loops (a curve like a distorted circle) in the counterclockwise way, and at time $t$, you are at the point $\{x[t], y[t]\}$. Does the unit normal vector

$$\frac{\{y'[t], -x'[t]\}}{\sqrt{x'[t]^2 + y'[t]^2}}$$

transplanted with its tail at $\{x[t], y[t]\}$ point out away from the curve toward your right foot, or does it point inside the curve toward your left foot?

R.8) Calculate the gradient, $\nabla f[x, y] = \text{grad}[x, y]$, of

$$f[x, y] = e^{-3xy}.$$ 

Calculate the gradient, $\nabla f[x, y] = \text{grad}[x, y, z]$, of

$$f[x, y, z] = x \sin[x\ y\ z].$$

R.9) Go with

$$f[x, y] = \frac{0.6}{1 + 1.2x^2 + 1.4y^2},$$

and look at a plot of $\text{grad}[x, y]$ with tails at $\{x, y\}$ for some selected points $\{x, y\}$ in the vicinity of $\{0, 0\}$.
Why are these gradient vectors so attracted to the point \( \{0, 0\} \)?

\( \Box \text{R.10} \)

Go with

\[ f[x, y, z] = x y \sin[z]. \]

In which direction should you leave the point \( \{1, 2, 0\} \) to get the greatest possible initial increase of \( f[x, y, z] \)?

In which direction should you leave the point \( \{1, 2, 0\} \) to get the greatest possible initial decrease of \( f[x, y, z] \)?

\( \Box \text{R.11} \)

The point \( \{1, 1\} \) is on the level curve

\[ x^2 y^3 e^{x+y} = 7.38906; \]

Here's a plot of part of this level curve with axes going through the point \( \{1, 1\} \):

Pencil in a vector with its tail at \( \{1, 1\} \) that points in the same direction as
\[ \text{gradf}[1, 1]. \]

\( \square \text{R.12)} \)

Does
\[ f[x, y] = x^6 - 3 x^2 y + y^4 - 6 x + 5 y \]
have a maximizer or minimizer? How do you know?

Does
\[ f[x, y] = \frac{x^5 + 3 y^2}{1 + x^4 + y^6} \]
have a maximizer or a minimizer? How do you know?

\( \square \text{R.13)} \)

Calculate
\[ \frac{\partial f[x,y,z]}{\partial z} = f^{(0,0,1)}[x, y, z] \]
by hand for
\[ f[x, y, z] = \sin[x^2 y^3 z^4]. \]

\( \square \text{R.14)} \)

Here is a 2D region \( R \) whose left boundary is the curve
\[ \text{left}[y] = \sin[3 y] \]
and whose right boundary is the curve
\[ \text{right}[y] = 4 \sin[y]: \]
Say why it's natural to calculate
\[ \int \int_R 1 \, dx \, dy \]
by integrating with respect to \( x \) first.
Next, calculate
\[ \int \int_R 1 \, dx \, dy \]
by hand and say what
\[ \int \int_R 1 \, dx \, dy \]
measures.

\( \square \) **R.15**

Go with
\[ f[x, y] = 2 \, x + 5 \, \text{Sin}[y]. \]
Give a clean formula for the function \( n[x, y] \) defined by
\[ n[x, y] = \int_0^x f[s, y] \, ds. \]
How are \( f[x, y] \) and \( D[n[x, y], x] \) related?
Is the outcome an accident? Why or why not?

\( \square \) **R.16**

You are given that the point \( \{2, 1\} \) is on a certain trajectory in the vector field
\[ \text{Field}[x, y] = \{x + y, x - y\}. \]
Write down a vector that is tangent to this trajectory at the point \( \{2, 1\} \).
Here's a circle and the plot of a certain vector field Field[x, y] plotted with tails at {x, y} for a generous selection of points {x, y} on the circle.

Here are tangential and normal components of what you see above:

Look at these plots, and then estimate whether the net flow of this vector field across the circle is from inside to outside, or is from outside to inside, and estimate whether the net flow of this vector field along this circle is clockwise or counterclockwise.

Here is a rectangle C with eight labeled points:
Go with the vector field 

\[ \text{Field}[x, y] = \{ y + x, -x \} \].

For each labeled point \( \{x, y\}\), pencil in the field vector \( \text{Field}[x, y] \) with tail at \( \{x, y\} \).

On the basis of your plot, estimate whether the net flow of this vector field across the edge of this rectangle is from inside to outside, or is from outside to inside, and estimate whether the net flow of this vector field along this curve is clockwise or counterclockwise.

\[ \square \hspace{1cm} \text{R.19} \hspace{1cm} \]

Suppose \( f[x_0, y_0] > f[x, y] \) for all other \( \{x, y\} \) near \( \{x_0, y_0\} \). If you center a small circle \( C \) at \( \{x_0, y_0\} \), why do you expect that the net flow of \( \text{grad} f[x, y] \) across \( C \) is from outside to inside?

Suppose \( f[x_0, y_0] < f[x, y] \) for all other \( \{x, y\} \) near \( \{x_0, y_0\} \). If you center a small circle \( C \) at \( \{x_0, y_0\} \), why do you expect that the net flow of \( \text{grad} f[x, y] \) across \( C \) is from inside to outside?
R.20)
You have a given vector field
\[ \text{Field}[x, y] = \{m[x, y], n[x, y]\}. \]
You have parameterized the ellipse
\[ \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \]
in the counterclockwise way with
\[ \{x[t], y[t]\} = \{2 \cos[t], 3 \sin[t]\} \]
with \(0 \leq t \leq 2 \pi\).
You calculate
\[ \oint_C m[x, y] \, dx + n[x, y] \, dy \]
\[ = \int_0^{2\pi} (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) \, dt, \]
and learn that
\[ \oint_C m[x, y] \, dx + n[x, y] \, dy = 9.01. \]
Then you calculate
\[ \oint_C -n[x, y] \, dx + m[x, y] \, dy \]
\[ = \int_0^{2\pi} (-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]) \, dt, \]
and learn that
\[ \oint_C -n[x, y] \, dx + m[x, y] \, dy = -3.52. \]
Is the net flow of Field[x, y] along C clockwise or counterclockwise?
Is the net flow of Field[x, y] across C from outside to inside, or is it from inside to outside?

R.21)
Measure by hand calculation the net flow across the circle
\[ x^2 + y^2 = 4 \]
of a fluid whose velocity is given by the vector field
\[ \text{Field}[x, y] = \{x, y\}. \]
Don't forget that \(\sin[t]^2 + \cos[t]^2 = 1\).
Is the net flow of F[x, y] across this curve from inside to outside or outside to inside?
Measure by hand calculation the net flow along the circle

\[ x^2 + y^2 = 9 \]

of a fluid whose velocity is given by the vector field

\[ \text{Field}[x, y] = \{-y, x\}. \]

Don't forget that \( \sin^2 t + \cos^2 t = 1 \).

Is the net flow of \( \text{F}[x, y] \) along this curve clockwise or counterclockwise?

□ **R.22**

Suppose \( C_1 \) and \( C_2 \) are physically the same curve, but they are parameterized so that the starting point of \( C_1 \) is the ending point of \( C_2 \), and the ending point of \( C_1 \) is the starting point of \( C_2 \).

Express

\[ \int_{C_1} m[x, y] \, dx + n[x, y] \, dy \]

in terms of

\[ \int_{C_2} m[x, y] \, dx + n[x, y] \, dy. \]

□ **R.23**

Take a given vector field

\[ \text{Field}[x, y] = \{m[x, y], n[x, y]\}. \]

If \( C_1 \) and \( C_2 \) are two curves that are parameterized to start at a given point \( \{a, b\} \), and end at another given point \( \{c, d\} \), but are not the same physical curve, what do you look for to determine whether you guaranteed that

\[ \int_{C_1} m[x, y] \, dx + n[x, y] \, dy = \int_{C_2} m[x, y] \, dx + n[x, y] \, dy? \]

□ **R.24**

Here are four vector fields:

\[ \begin{align*}
\text{Field1}[x, y] &= \{y \, e^{x \, y}, x \, e^{x \, y}\}, \\
\text{Field2}[x, y] &= \{y, -x\}, \\
\text{Field3}[x, y] &= \{\sin[2 \, x], \cos[5 \, y]\}, \text{ and} \\
\text{Field4}[x, y] &= \{3 \, x - 2 \, y, -2 \, x + 5 \, y\}.
\end{align*} \]

All but one of these vector fields are gradient fields. Identify the oddball.
R.25)
Does 
Field[x, y] = {y Sin[x y], x Sin[x y]}
pass the gradient test?
Does 
Field[x, y] = {x Sin[x y], y Sin[x y]}
pass the gradient test?

R.26)
Go with 
Field[x, y] = {e^x Sin[y], e^x Cos[y]}
and calculate divField[x, y].
What does your result tell you about the net flow of Field[x, y]
across the ellipse \((\frac{x}{3})^2 + (\frac{y}{7})^2 = 1\)?

R.27)
Here's the rectangle R with corners at 
{-2, 0}, {3, 0}, {3, \pi} and {-2, \pi}: 

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Use a 2D integral to measure the net flow of the vector field

\[
\text{Field}[x, y] = \{x \cos[y], e^{-x^2/2} + y^2\}
\]

across the boundary curve \( C \) of this rectangle.

Say why you are happy to make this measurement by calculating your 2D integral instead of making this measurement by calculating the path integral

\[
\oint_C - (e^{-x^2/2} + y^2) \, dx + x \cos[y] \, dy
\]

\[
\text{R.28) }
\]

Identify the points \( \{x, y\} \) of the vector field

\[
\text{Field}[x, y] = \{-x^2 + y, y + 2 \sin[\pi x]\}
\]

that are sources of new fluid.

Identify the points \( \{x, y\} \) of the vector field

\[
\text{Field}[x, y] = \{-x^2 + y, y + 2 \sin[\pi x]\}
\]

that are sinks for old fluid.

\[
\text{R.29) }
\]

Given a vector field

\[
\text{Field}[x, y] = \{m[x, y], n[x, y]\}
\]

with the extra property that

\[
\text{divField}[x, y] = D[m[x, y], x] + D[n[x, y], y] > 0
\]

for every \( x \) and \( y \), explain how you know that the net flow of

\[
\text{Field}[x, y] = \{m[x, y], n[x, y]\}
\]

across any closed curve without loops is from inside to outside.

What happens in the case that \( \text{divField}[x, y] < 0 \) for every \( x \) and \( y \)?

What happens in the case that \( \text{divField}[x, y] = 0 \) for every \( x \) and \( y \)?