ELLIPTIC CURVES AND ALGEBRAIC TOPOLOGY

MATTHEW ANDO

Part 1. Elliptic curves and chromatic stable homotopy theory

Elliptic curves enter algebraic topology through "Elliptic cohomology"–really a family of cohomology theories–and their associated "elliptic genera".

- Arithmetic aspect: Modularity of elliptic genera, The spectrum TMF of "topological modular forms" and the calculation of $\pi_* TMF \to MF(\mathbb{Z})$, Hopkins's proof of Borcherds' congruences.
- Physical aspect: Witten's approach to elliptic genera via string theory.
- Homotopy theoretic aspect: Relationship to chromatic program, Hopkins and Mahowald's calculation of $\pi_* S \to \pi_* TMF$.

1. The Adams spectral sequence and descent

Forgetful functor
Not quite faithful. Instead
Adams Spectral Sequence

Let E be a generalized (co)homology theory. One could hope to study maps from X to Y by studying maps

$$E_* X \to E_* Y$$

(of $E_* E$-comodules).

For example, consider

$$S^1 \xrightarrow{n} S^1$$

This is detected perfectly well by $E = H\mathbb{Z}$; when $E = H\mathbb{Z}/2$, at least it captures the parity of n.

For $E = H\mathbb{Z}/2$, $n = 2$, one uses the cofiber sequence

$$S^1 \xrightarrow{2} S^1 \to M = e^2 \cup_2 S^1 \to S^2;$$

this gives a short exact sequence

$$E_* S^1 \to E_* M \to E_* S^2$$

i.e. an element of

$$\mathrm{Ext}^1_{E_* E}(E_* S^1, E^* S^2) = \mathrm{Ext}^{1,1}_{E_* E}(E_* S^0, E_* S^0)$$

Note that considering $E_* E$ comodules distingushes M from $S^1 \vee S^2$, which corresponds to the zero map $S^1 \to S^1$.

The *Adams spectral sequence* is of the form

$$E_2 = \mathrm{Ext}^{s,t}_{E_* E}(E_* S^0, E_* X) \Rightarrow \pi_{t-s} L_E X.$$

The exact couple can be obtained from the sequence

$$S \longrightarrow E \rightrightarrows E \wedge E \Rrightarrow E \wedge E \wedge E \dots.$$

This is analogous to the sequence

$$S \longrightarrow E \rightrightarrows E \otimes_S E \Rrightarrow E \otimes_S E \otimes_S E$$

for faithfully flat descent: if $S \to E$ is a faithfully flat map of commutative rings, then the forgetful functor

$$(S\text{-modules}) \to (E\text{-modules with descent data})$$

is an equivalence of categories.

Date: July 2003.

After applying homotopy groups, one has

$$\pi_* S \longrightarrow \pi_* E \rightrightarrows E_* E \Rrightarrow E_* E \otimes_{E_*} E_* E$$

This is an example of a "Hopf algebroid". Let $A = E_*$ and $\Gamma = E_* E$; then the basic object of study is

$$A \rightrightarrows \Gamma \Rrightarrow \Gamma \otimes_A \Gamma$$

2. Geometric reformulation

A Mittag-Leffler sequence is a sequence of epimorphisms.–A. Neeman.

In the language of algebraic geometry, a Hopf algebroid is a group acting on a space

$$G \times G \times T \Rrightarrow G \times T \underset{\pi}{\overset{\mu}{\rightrightarrows}} T \dashrightarrow T/G$$

and a comodule is an equivariant sheaf. That is

$$\begin{aligned} \operatorname{sp} E_* &= T \\ \operatorname{sp} E_* E &= G \times T \\ \widetilde{E_* X} &= V; \end{aligned}$$

and the E_2-term of the Adams spectral sequence is just

$$\operatorname{Ext}^*_{E_* E}(E_*, E_* X) = H^*(G; V),$$

the G-equivariant cohomology of the sheaf V, which is the natural object to study for questions of descent to T/G.

3. The Adams-Novikov spectral sequence

Suppose that E is *complex oriented*: that is, it comes with an isomorphism

$$E^* \mathbb{C}P^\infty \cong E^*[\![x]\!],$$

and so the map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

classifying the tensor product of complex line bundles gives rise to a map

$$E^*[\![x]\!] \cong E^* \mathbb{C}P^\infty \to E^* \mathbb{C}P^\infty \times \mathbb{C}P^\infty \cong E^*[\![x, y]\!],$$

that is, a *formal group law* over E.

The initial example of a complex oriented cohomology theory is MU, complex cobordism. Quillen showed that $\pi_* MU$ carries the *universal* formal group law; and it turns out that $MU_* MU$ represents the group of isomorphisms of formal group laws. That is,

$$\begin{aligned} \operatorname{sp} \pi_* MU &= FGL \\ \operatorname{sp} MU_* MU &= FGLI \times FGL. \end{aligned}$$

Thus the E_2-term of the MU-Adams spectral sequence ("Adams-Novikov spectral sequence") for X is

$$\operatorname{Ext}_{MU_* MU}(\pi_* MU; MU_* X) \cong H^*(FGLI; \widetilde{MU_* X});$$

it's calculating descent for

$$FGLI \times FGL \underset{\pi}{\overset{\mu}{\rightrightarrows}} FGL \dashrightarrow \text{(formal groups)}$$

Morava took this picture seriously, and Devinatz, Hopkins, Landweber, Miller, Ravenel, Smith, Wilson…took him seriously, and so we have chromatic stable homotopy theory. Note that this is saying that the differentials in the Adams-Novikov spectral sequence are measuring the difference between the (homotopy groups of) spheres and (the functions on) the space of formal groups.

Locally at a prime p, a formal group has an important invariant called the *height*: it's a nonnegative integer. Associated to each height is a family of cohomology theories which detect the associated chromatic homotopy theory.

height	0	1	2	n
Morava	$H\mathbb{Q}$	K/p	Ell/p	$K(n)$
Lubin-Tate	$H\mathbb{Q}$	K_p	Ell_p	E_n
Global	$H\mathbb{Q}$	K	Ell	?
Formal group	$\mathbb{G}_a$	$\mathbb{G}_m$	$\widehat{C}$	Honda

For example, localizing to a neighborhood of a height n formal group Γ replaces MU with E_n and $FGLI$ with $\mathbb{S}_n = Aut(\Gamma)$, and the Adams spectral sequence becomes

$$H^*(\mathbb{S}_n; E_{n*}X) \Rightarrow \pi_* L_{K(n)}X.$$

For $n = 1$ this spectral sequence is well understood and describes the part of π_*S detected by the J-homomorphism. For $n \geq 2$ the situation is more complicated.

4. Elliptic cohomology

An *elliptic cohomology theory* is a commutative, even periodic cohomology theory, whose formal group is given with an isomorphism to the formal group of an elliptic curve. The formal groups of (generalized) elliptic curves have height (0 and) 1 and 2.

Two big advantages of elliptic cohomology (over E_2, say).

1) Elliptic curves are projective algebraic objects; for example this helps with calculation: for example, the sigma orientation and the calculation by Hopkins and Mahowald of $\pi_*TMF \to MF(\mathbb{Z})$.

2) Elliptic cohomology appears to offer a relationship to geometry and analysis.

5. What is TMF?

The definition of elliptic cohomology suggests the question: which elliptic curves admit elliptic cohomolgy theories? Can the construction be made functorial?

Hopkins and Miller have constructed a functor

$$\mathbf{C} : (\text{étale elliptic curves}) \to (E_\infty \text{ ring spectra})$$

The spectrum of *topological modular forms* is the homotopy inverse limit

$$TMF = \operatorname{holim} \mathbf{C}.$$

It turns out that TMF is not an elliptic spectrum, but by construction there is a canonical map from TMF to any E_∞ elliptic spectrum.

In order to give moduli for the space of formal groups, it was necessary to describe the space of formal groups as the quotient

$$(\text{formal groups}) \cong FGL/FGLI.$$

This is also the case for the space of elliptic curves: every elliptic curve is isomorphic to a Weierstrass curve, i.e. one of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

and two Weierstrass cubics give the same elliptic curve if they differ by

$$x' = u^2x + ry' = u^3y + sx + t.$$

The space of elliptic curves is the quotient

$$ELL \cong W/WI.$$

The dual Lie algebra of the universal elliptic curve over W defines an equivariant sheaf ω over W. By construction there is a spectral sequence with

$$E_2 = H^s(WI; \omega^t) \Rightarrow \pi_{t-s}TMF.$$

The name "topological modular forms" comes from the fact that

$$H^0(WI; \omega^t)$$

is the group of integral modular forms of weight $\pm t$. Indeed the edge homomorphism gives a map

$$\pi_* TMF \to MF_*(\mathbb{Z}).$$

Hopkins and Mahowald have calculated this map completely.

6. E_∞ haiku

Cochain cup product
Not quite commutative
Steenrod operations

7. Geometric and Physical Aspect

In any cohomology theory E one might try to study the *Umkehr* map. If $f : X \to Y$ is a proper map with fiber dimension d, then this is a homomorphism

$$f_! : E^* X \to E^{*-d} Y.$$

The construction depends on the choice of an orientation in E-theory for the bundle T_f of tangents along the fibers of f. An important special case is the map

$$\pi^X : X \to *;$$

by applying $\pi_!$ to various characteristic classes of X you get various "genera" associated to E-theory.

The choice of orientation is really the crux of the matter: for example, a consistent family of orientations in E-theory for spin vector bundles is equivalent to a map of ring spectra

$$MSpin \to E,$$

and the ring homomorphism

$$\pi_* MSpin \to \pi_* E$$

sends X to $\pi_!^X(1)$.

The interpretation of this map often offers insight into the geometry and analysis associated to E. For example

H	integration along the fiber
K	index of family of elliptic operators
Ell	one-loop amplitude of string theories.

The calculation of elliptic genera using string theory is more than an idle observation: it leads to physical proofs of mathematical results: particularly, the modularity and rigidity of the Witten genus

$$w : \pi_* MSpin \to \mathbb{Z}[\![q]\!].$$

Let M be a spin manifold with Dirac operator D. Let T be its tangent bundle. Let V be another spin vector bundle, and let $\Delta_{-1} V$ be the associated spinor bundle. Let

$$S_t : KO(X) \to K(X)[\![t]\!]$$

be the exponential characteristic class

$$S_t W = \sum_{k \geq 0} t^k S^k (W \otimes \mathbb{C})$$

and similarly for Λ. If W is a vector bundle, let $\overline{W}$ be the associated reduced bundle. Then

$$w(M; V) = \\ \operatorname{ind}(D \otimes \bigotimes_{n \geq 0} S_{q^n} \overline{T} \otimes \Delta_{-1} V \otimes \bigotimes_{n \geq 0} \Lambda_{q^n} \overline{V}) \\ \in \mathbb{Z}[\![q]\!].$$

Witten gave physical proofs of two results about w.

Proposition (Modularity). *If $c_2(T - V) = 0$, then $w(M; V)$ is the q-expansion of a modular form.*

If S^1 acts on the whole situation, then we can consider the equivariant Witten genus

$$w_{S^1}(M;V) \in (\mathbb{Z}[\lambda, \lambda^{-1}])[\![q]\!].$$

Proposition (Rigidity)**.** *If $w_2(T-V)_{S^1} = 0$ and $c_2(T-V)_{S^1} = 0$, then for all k the coefficient of q^k in $w_{S^1}(M;V)$ is a constant Laurent polynomial; that is,*

$$w_{S^1}(M;V) = w(M;V).$$

To explain the modularity of the Witten genus, we locate it in elliptic cohomology. To begin with, there is an elliptic cohomology theory based on the Tate elliptic curve; it was discovered by Morava in the early 70's. Because the formal group of the Tate curve is the multiplicative formal group, it is a form of K-theory, with coefficients extended to power series in q.

Hopkins, Rezk, and I have constructed a map

$$\sigma : MO\langle 8\rangle \to TMF,$$

such that the diagram

$$\begin{tikzcd} MO\langle 8\rangle \arrow[r] \arrow[d] & TMF \arrow[d] \\ MSpin \arrow[r, "w"] & K[\![q]\!] \end{tikzcd}$$

commutes.

It follows that the Witten genus of an $MO\langle 8\rangle$ manifold is the q-expansion of a modular form. Indeed it is in the image of

$$\pi_* tmf \to MF(\mathbb{Z}) \to \mathbb{Z}[\![q]\!].$$

Hopkins and Mahowald have shown that this map has a cokernel, which corresponds to divisibility results for Witten genera.

Similarly, the kernel (which is all 24-torsion) gives new torsion invariants for $MO\langle 8\rangle$ manifolds.

Part 2. Modularity in elliptic cohomology

In the previous lecture I introduced an invariant of spin manifolds called the *Witten genus.*

Let M be a spin manifold with Dirac operator D. Let T be its tangent bundle. Let V be another spin vector bundle, and let $\Delta_{-1}V$ be the associated spinor bundle. Let

$$S_t : KO(X) \to K(X)[\![t]\!]$$

be the exponential characteristic class

$$S_t W = \sum_{k\geq 0} t^k S^k (W \otimes \mathbb{C})$$

and similarly for Λ. If W is a vector bundle, let $\overline{W}$ be the associated reduced bundle. Then

$$w(M;V) =$$
$$\operatorname{ind}(D \otimes \bigotimes_{n\geq 1} S_{q^n}\overline{T} \otimes \Delta_{-1}V \otimes \bigotimes_{n\geq 1} \Lambda_{q^n}\overline{V})$$
$$\in \mathbb{Z}[\![q]\!].$$

Taking $V = 0$ gives the Witten genus; it is a ring homomorphism

$$w : \pi_* MSpin \to \mathbb{Z}[\![q]\!]$$

Witten gave physical proofs of two results about w.

Proposition (Modularity)**.** *If $c_2(T-V) = 0$, then $w(M;V)$ is the q-expansion of a modular form.*

If S^1 acts on the whole situation, then we can consider the equivariant Witten genus

$$w_{S^1}(M;V) \in (\mathbb{Z}[\lambda, \lambda^{-1}])[\![q]\!].$$

Proposition (Rigidity)**.** *If $w_2(T-V)_{S^1}=0$ and $c_2(T-V)_{S^1}=0$, then for all k the coefficient of q^k in $w_{S^1}(M;V)$ is a constant Laurent polynomial; that is,*

$$w_{S^1}(M;V)=w(M;V).$$

I explained that the modularity of the Witten genus is an easy consequence of the existence of a map of ring spectra $\sigma : MO\langle 8\rangle \to TMF$ such that the diagram

$$\begin{array}{ccc} MO\langle 8\rangle & \xrightarrow{\sigma} & TMF \\ \downarrow & & \downarrow \\ MSpin & \xrightarrow{w} & > KO \end{array}$$

commutes.

In this lecture I shall explain how to construct a natural map

$$\sigma(E,C,t) : MU\langle 6\rangle \to E$$

from the bordism spectrum of stably complex manifolds with trivialization of c_1 and c_2 to any elliptic spectrum, such that the diagram

$$\begin{array}{ccc} MU\langle 6\rangle & \xrightarrow{\sigma(K_{\mathrm{Tate}})} & K[\![q]\!] \\ \downarrow & & \uparrow w \\ MSU & \longrightarrow & MSpin \end{array}$$

commutes. Already this proves that the Witten genus of any $BU\langle 6\rangle$ manifold is the q-expansion of a modular form.

The proof will eventually reduce to the observation that if

$$\sigma(L,q)=(L^{1/2}-L^{-1/2})\prod_{n\geq 1}\frac{(1-q^nL)(1-q^nL^{-1})}{(1-q^n)^2},$$

then on the one hand $\sigma(L,q)$ is just the product formula for the Weierstrass sigma function, and on the other hand if we write $T=TM$ as a sum of complex line bundles

$$T=L_1+\cdots+L_r,$$

then

$$\Delta_{-1}(T)\otimes\bigotimes_{n\geq 1}S_{q^n}(\overline{T})\cong\prod_{j=1}^{r}\sigma(L_i,q)^{-1}.$$

8. Genera after Hirzebruch

Let R be a $\mathbb{Q}$-algebra. To give a ring homomorphism

$$MU_* \xrightarrow{\phi} R,$$

it is equivalent to give a power series

$$f(x)=x+o(x^2)\in R[\![x]\!].$$

If ϕ is such a genus, then the associated power series is determined by

$$f(x)^{-1}=\sum_{n\geq 0}\frac{\phi(\mathbb{C}P^n)}{n+1}x^{n+1}. \tag{1}$$

Let f be a power series. If M is a stably complex manifold of dimension $2r$, and if we use the splitting principle to write

$$TM=L_1+\cdots+L_r,$$

and set $x_i=c_1L_i$, then

$$\phi(M)=\int_M\prod_{j=1}^{r}\frac{x_i}{f(x_i)}.$$

For example, the genus assoicated to the power series

$$f(x) = 1 - e^{-x}$$

is called the Todd genus; equation (1) reflects the fact that $\mathrm{Todd}(\mathbb{C}P^n) = 1$ for all n.

9. Genera after Dold and Adams

Let E be a commutative ring spectrum. Then maps of ring spectra

$$MU \to E$$

are in bijective correspondence with classes

$$u \in E^2(\mathbb{C}P^\infty)$$

such that

$$u|_{\mathbb{C}P^1} = \Sigma^2(1)$$

in $E^2(\mathbb{C}P^1)$.

Given such a u, the associated genus

$$\phi : \pi_* MU \to \pi_* E$$

is given by the formula

$$\phi(M) = \pi_!^M(1),$$

where $\pi_!^M$ is the Umkehr map

$$E^*(M) \to E^{*-d}(*)$$

associated to the map $\pi^M : M \to *$ and the orientation u.

Consider

$$E^*(\mathbb{C}P^\infty) \xrightarrow{c} (E \wedge H\mathbb{Q})^*(\mathbb{C}P^\infty) \leftarrow H\mathbb{Q}^*(\mathbb{C}P^\infty).$$

If $x = c_1 L$, then we can write

$$c(u) = f(x) \in (E \wedge H\mathbb{Q})^*[\![x]\!].$$

The Riemann-Roch formula is

$$\pi_!^M(1) = \int_M \prod_j \frac{x_j}{f(x_j)},$$

where x_j are the roots of the total Chern class of M.

For example, if $E = K$ and $v \in K^0(\mathbb{C}P^1) \cong K^{-2}(*)$ is the Bott element, then

$$c : K \to K \wedge H\mathbb{Q} \cong H\mathbb{Q}[v, v^{-1}]$$

is the Chern character, and

$$u = v^{-1}(1 - L) \in K^2(\mathbb{C}P^\infty)$$

gives an orientation

$$MU \to K,$$

such that

$$c(u) = v^{-1}(1 - e^{vx}),$$

and so

$$\pi_!^M(1) = v^r \,\mathrm{Todd}(M).$$

where $\dim M = 2r$.

10. Dold-Adams II

Let L be the tautological line bundle over $\mathbb{C}P^\infty$. The map $\mathbb{C}P^\infty \to BU$ classifying the reduced bundle $L-1$ gives a map of Thom spectra

$$\Sigma^{-2}(\mathbb{C}P^\infty)^L \cong (\mathbb{C}P^\infty)^{L-1} \xrightarrow{j} MU.$$

Restricting to the basepoint of $\mathbb{C}P^\infty$ gives

$$S^0 \to (\mathbb{C}P^\infty)^{L-1} \xrightarrow{j} MU.$$

Proposition. *If E is a commutative ring spectrum, then restriction along j gives an isomorphism*

$$\mathrm{RingSpectra}(MU, E) \cong \pi_0 \left(\begin{tikzcd} (\mathbb{C}P^\infty)^{L-1} \arrow[r, dashed] & E \\ S^0 \arrow[u] \arrow[ur, "\eta"'] & \end{tikzcd} \right)$$

The homeomorphism $(\mathbb{C}P^\infty)^L \cong \mathbb{C}P^\infty$ identifies the right-hand side with the set of $u \in E^2(\mathbb{C}P^\infty)$ such that $u|_{\mathbb{C}P^1} \cong \Sigma^2(1)$.

11. Algebro-geometric formulation

Let E be a commutative, even periodic ring spectrum. Then $E^0\mathbb{C}P^\infty$ is the ring of functions on a formal group G over $S_E = \mathrm{sp}\,\pi_0 E$. The augmentation

$$E^0\mathbb{C}P^\infty \to \pi_0 E$$

corresponds to the identity section $S_E \xrightarrow{0} G$.

The zero section

$$\mathbb{C}P^\infty \to (\mathbb{C}P^\infty)^L$$

identifies $E^0((\mathbb{C}P^\infty)^L)$ with the ideal $I_G(0)$ of functions on G which vanish at 0. The restriction

$$E^0((\mathbb{C}P^\infty)^L) \to E^0 S^2$$

identifies $E^0S^2 \cong \pi_0 E$ with

$$\omega \stackrel{\text{def}}{=} 0^* I_G(0) \cong T_0^* G,$$

the cotangent space of G at the origin, and so $E^0 S^{-2} \cong \omega^{-1}$.

Thus

$$E^0((\mathbb{C}P^\infty)^{L-1})$$

is the $E^0\mathbb{C}P^\infty$-module of sections of $I_G(0) \otimes p^*\omega^{-1}$. Notice that

$$0^*(I_G(0) \otimes p^*\omega^{-1}) \cong \omega \otimes \omega^{-1}$$

has a canonical trivialization; this corresponds to the fact that

$$E^0(\Sigma^{-2}S^2) \cong E^0(S^0) \ni 1.$$

Proposition. *The set of maps of ring spectra $MU \to E$ is in bijective correspondence with the set of rigid sections of $I_G(0) \otimes p^*\omega^{-1}$.*

12. MSU and $MU\langle 6\rangle$

Let V_k be the virtual bundle

$$V_k = \prod_{j=1}^{k}(1 - L_j)$$

over $(\mathbb{C}P^\infty)^k$. It turns out that the map

$$(\mathbb{C}P^\infty)^k \to BU$$

classifying this bundle factors uniquely through $BU\langle 2k\rangle$, and so we get a map of Thom spectra

$$(\mathbb{C}P^{\infty k})^{V_k} \to MU\langle 2k\rangle.$$

It turns out that $E^0((\mathbb{C}P^{\infty k})^{V_k}))$ is the module of sections of $\Theta^k(I_G(0))$. Θ^k is most easily illustrated by example: if $\mathcal{L}$ is a line bundle over G, then $\Theta^k(\mathcal{L})$ is a line bundle over G^k given by

$$\begin{aligned}
\Theta^0(\mathcal{L}) &= \mathcal{L} \\
\Theta^1(\mathcal{L})_a &= \frac{\mathcal{L}_a}{\mathcal{L}_0} \\
\Theta^2(\mathcal{L})_{a,b} &= \frac{\mathcal{L}_{a+b}\mathcal{L}_0}{\mathcal{L}_a\mathcal{L}_b} \\
\Theta^3(\mathcal{L})_{a,b,c} &= \frac{\mathcal{L}_{a+b+c}\mathcal{L}_a\mathcal{L}_b\mathcal{L}_c}{\mathcal{L}_{a+b}\mathcal{L}_{a+c}\mathcal{L}_{b+c}\mathcal{L}_0}.
\end{aligned}$$

Notice that for $k \geq 1$, $\Theta^k(\mathcal{L})$ is

(1) *rigid*: there is a canonical isomorphism

$$\Theta^k(\mathcal{L})_0 \cong \mathbf{1}$$

(2) *symmetric*: for $\sigma \in \Sigma_k$, there is a canonical isomorphism

$$\sigma^*\Theta^k(\mathcal{L}) \cong \Theta^k(\mathcal{L})$$

In addition, for $k \geq 2$, Θ^k satisfies a cocycle condition. A Θ^k-structure on $\mathcal{L}$ is a trivialization of $\Theta^k(\mathcal{L})$ which is a rigid symmetric cocycle.

Theorem. *Restriction to $(\mathbb{C}P^{\infty k})^{V_k}$ gives a map*

$$\mathrm{RingSpectra}(MU\langle 2k\rangle, E) \to (\Theta^k\textit{-structures on } I_G(0)).$$

For $1 \leq k \leq 3$, this is an isomorphism.

Suppose that we've chosen an isomorphism

$$E^0\mathbb{C}P^\infty \cong E^0[\![x]\!].$$

Then for $k = 3$, the theorem identifies the set of maps of ring spectra

$$MU\langle 6\rangle \to E$$

with the set of power series

$$f(x, y, z) \in E^0[\![x, y, z]\!]$$

which are symmetric in x, y, x and satisfy

$$\begin{aligned}
f(x, y, z) &= xyz + \text{higher terms} \\
f(x, y, z)f(w, x + y, z) &= f(w + x, y, z)f(w, x, z),
\end{aligned}$$

where the sum inside parentheses uses the group law of $E^0\mathbb{C}P^\infty$. The set of such f are called *cubical structures* on $I_G(0)$.

To calculate the corresponding genus, find a power series $g(x) = x + o(x^2)$ such that

$$f(x, y, z) = \frac{g(x + y + z)g(x)g(y)g(z)}{g(x + y)g(x + z)g(y + z)}.$$

The genus is

$$\int_M \prod_j \frac{x_j}{g(x_j)}.$$

13. The sigma orientation

Theorem (Abel)**.** *Let C be an elliptic curve, that is, a pointed proper smooth curve whose geometric fibers have genus 1. Then C has a unique structure of group such that $I_C(0)$ has a cubical structure. Every map $C \to C'$ of elliptic curves is a homomorphism of groups.*

Now let (E, C, t) be an elliptic spectrum. Abel's Theorem gives a cubical structure $s(C)$ on $I_C(0)$, and $t^* s(C)|_{\widehat{C}}$ gives a cubical structure on $I_G(0)$, and so a map of ring spectra

$$MU\langle 6\rangle \to E.$$

Abel's Theorem is characterizes the group structure of an ellliptic curve by saying that given points y and z in C, there is a meromorphic function on C with divisor $(0) + (-y-z) - (-y) - (-z)$. If $C = \mathbb{C}/\Lambda$, then this function is a scalar multiple of

$$\frac{\sigma(x+y+z)\sigma(x)}{\sigma(x+y)\sigma(x+z)}.$$

The cube structure arises from symmetrizing this to get

$$\frac{\sigma(x+y+z)\sigma(x)\sigma(y)\sigma(z)}{\sigma(x+y)\sigma(x+z)\sigma(y+z)}.$$

This shows that, in the case of the Tate curve, the sigma orientation is the restriction of the Witten genus.

Part 3. Equivariant elliptic cohomology and rigidity

Let M be a spin manifold with tangent bundle T and Dirac operator D. Let V be a spin vector bundle of even rank over M. Suppose moreover that the circle S^1 acts on the whole situation. Then the equivariant Witten genus of M twisted by V is

$$\begin{aligned} w(M;V)_{S^1} = \operatorname{ind}_{S^1}(D \otimes & \bigotimes_{n\geq 1} S_{q^n}(\overline{T}) \otimes \\ & \Delta_{-1}(V) \otimes \\ & \bigotimes_{n\geq 1} \Lambda_{q^n}(\overline{V})) \in (R(S^1))[\![q]\!]. \end{aligned}$$

We fix an isomorphism $R(S^1) \cong \mathbb{Z}[\lambda, \lambda^{-1}]$. Thus

$$w(M;V)_{S^1} = \sum_k a_k(\lambda) q^k$$

is a power series in q, with each coefficient itself a Laurent polynomial. Witten gave a physical proof of the following

Theorem (Ridigity)**.** *If the equivariant characteristic classes $w_2(V-T)_{S^1}$ and $c_2(V-T)_{S^1}$ vanish, then each $a_k(\lambda)$ is constant, that is,*

$$w(M;V)_{S^1} = w(M;V).$$

The rigidity theorem was first proved by Bott and Taubes, and Kefeng Liu gave another proof. I shall explain how this result follows from the existence of a Thom isomorphism in Grojnowski's equivariant elliptic cohomology, extending the sigma orientation. The idea that there should be such a proof is due to Haynes Miller, and the first proof along these lines was given by Ioanid Rosu, although for the Ochanine genus, whose rigidity does not require any condition on c_2.

14. Equivariant elliptic cohomology

It is familiar that Atiyah-Segal completion can be written

$$\begin{array}{rcccccl} \operatorname{sp} K_{S^1}(*) & \cong & \mathbb{G}_m & \leftarrow & \widehat{\mathbb{G}}_m & \cong & \operatorname{spf} K(BS^1) \\ \operatorname{sp} K_T(*) & \cong & \check{T} \otimes \mathbb{G}_m & \leftarrow & \check{T} \otimes \widehat{\mathbb{G}}_m & \cong & \operatorname{spf} K(BT) \end{array}$$

$$\operatorname{sp} K_G(*) \cong (\check{T} \otimes \mathbb{G}_m)/W \leftarrow (\check{T} \otimes \widehat{\mathbb{G}}_m)/W \cong \operatorname{spf} K(BG).$$

In the second line, T is a torus and $\check{T} = \hom(S^1, T)$ is its lattice of cocharacteris; in the third line, G is a connected compact Lie group with maximal torus T and Weyl group W. One might hope that if (E, C, t) is an elliptic spectrum, then there is an equivariant elliptic cohomology with a completion map whose values are given by the table

$$\begin{array}{rcccccl} E_{S^1}(*) & \cong & C & \leftarrow & \widehat{C} & \cong & \operatorname{spf} E(BS^1) \\ E_T(*) & \cong & \check{T} \otimes C & \leftarrow & \check{T} \otimes \widehat{C} & \cong & \operatorname{spf} E(BT) \end{array}$$

$$E_G(*) \cong (\check{T} \otimes C)/W \leftarrow (\check{T} \otimes \widehat{C})/W \cong \operatorname{spf} E(BG).$$

On the left, I've omitted the sp and viewed E_G as a covariant functor from spaces to schemes.

This approach to equivariant elliptic cohomology was suggested by Ginzburg-Kapranov-Vasserot and Grojnowski. In fact Grojnowski has constructed a contravariant functor

$$E_{S^1} : (S^1\text{-spaces}) \to (\text{sheaves of } \mathcal{O}_C\text{-algebras})$$

when C is the analytic elliptic curve $\mathbb{C}/(2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z})$. It has the following properties.

(1) A completion isomorphism
$$E_{S^1}(X)^\wedge_0 \cong E(X_{S^1}).$$
(2) If X is a spin manifold, then $E_{S^1}(X^T)$ is an invertible $E_{S^1}(X)$-module, and there is a Pontryagin-Thom map
$$E_{S^1}(X^{-T}) \overset{\text{def}}{=} E_{S^1}(X^T)^{-1} \to E_{S^1}(*) = \mathcal{O}_C$$
which is compatible with the completion isomorphism.

Now consider the diagram

$$\begin{tikzcd} \Gamma E_{S^1}(X^{V-T}) \arrow[r] \arrow[d] & E(V_{S^1} - T_{S^1}) \arrow[d] \\ \Gamma E_{S^1}(X^{-T}) \arrow[r] \arrow[d] & E(-T_{S^1}) \arrow[d] \\ \Gamma E_{S^1}(*) \arrow[r] & E(BS^1) \end{tikzcd}$$

Now $E(V_{S^1} - T_{S^1})$ contains a class $W(M;V)$ which pushes forward to the Witten genus $w(M;V)_{S^1}$. On the other hand, the bottom left is just $\Gamma\mathcal{O}_C = \mathbb{C}$. Thus if we can construct a class $\gamma(M;V)$ in $\Gamma E_{S^1}(X^{V-T})$ which maps to $W(M;V)$ under the top horizontal map, i.e. such that $\gamma(M;V)_0$ is $W(M;V)$, then the Witten genus is rigid as required.

Theorem. *Let V and T be equivariant spin vector bundles of even rank over a S^1-space M. Suppose that $w_2(V-T)_{S^1}$ and $c_2(V-T)_{S^1}=0$. Then S^1-orientations on V and T determine a trivialiation $\gamma(T;V)$ of $E_{S^1}(V)\otimes E_{S^1}(T)^{-1}$ as $E_{S^1}(X)$-module, such that $\gamma(T;V)_0 = W(T;V)$. Moreover the association $(T;V)\mapsto\gamma(T;V)$ is exponential under Whitney sum and natural under pull-back.*

(A S^1-orientation on V is a choice of orientation on the fixed subbundle V^A/M^A for each closed subgroup A of S^1; it turns out that any equivariant spin bundle is orientable).

Thus γ is a sort of analytic equivariant sigma orientation

$$MO\langle 8\rangle_{S^1}\to E_{S^1}.$$

I shall give a conceptual proof of the theorem which also sheds light on the nonequivariant sigma orientation. In order to do so it is useful to explain how theta functions arise in connection with degree-four characteristic classes.

15. c_2 and theta functions

Let G be a connected compact Lie group with maximal torus T and Weyl group W. Let $\widehat{T}$ and $\check{T}$ be as usual the lattices of characters and cocharacters of T. Then we have a map

$$\begin{aligned} H^4(BG;\mathbb{Z})\to H^4(BT;\mathbb{Z})^W &\cong \operatorname{Sym}^2\widehat{T}^W \\ &\cong \hom(\Gamma_2\check{T},\mathbb{Z})^W. \end{aligned}$$

As I learned from Bill Dwyer, the arrow is an isomorphism if G is simply connected.

Thus a degree-four characteristic class c gives rise to a homomorphism $c:\Gamma_2\check{T}\to\mathbb{Z}$. Using c, define maps I and γ as follows.

$$\begin{tikzcd} \check{T}\otimes\check{T} \arrow[r] \arrow[rr, bend left, "I"] & \Gamma_2\check{T} \arrow[r, "c"'] & \mathbb{Z} \\ & \check{T} \arrow[u, "\gamma_2"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

I is bilinear, and ϕ is quadratic; they are related by the formula

$$\phi(u+v)=\phi(u)+I(u,v)+\phi(v).$$

If M is an abelian group written multiplicatively and A is an abelian group written additively, then the rules for manipulation in $A\otimes M$ are most easily remembered if $a\otimes m$ is written m^a. With this convention, suppose that $0<|q|<1$ and $C=\mathbb{C}^\times/q^{\mathbb{Z}}$. Then

$$\check{T}\otimes C\cong(\check{T}\otimes\mathbb{C}^\times)/z\sim zq^a, a\in\check{T}.$$

Given ϕ and I defined by a degree-four characteristic class c, consider the line bundle over $\check{T}\otimes C$ given by the formula

$$\mathcal{L}(c)=\frac{\check{T}\otimes\mathbb{C}^\times\times\mathbb{C}}{(z,\lambda)\sim(zq^a,q^{\phi(a)}z^{\widehat{I}(a)}\lambda)}.$$

The W-invariance of c implies that $\mathcal{L}(c)$ is a W-equivariant line bundle over $\check{T}\otimes C$, and so it descends to a line bundle $\mathcal{A}(c)$ over $(\check{T}\otimes C)/W$. The sections of $\mathcal{A}(c)$ are precisely the W-invariant sections of $\mathcal{L}(c)$; these are holomorphic functions

$$\theta=\theta(z,q):\check{T}\otimes\mathbb{C}^\times\to\mathbb{C}$$

such that

$$\begin{aligned} \theta(zq^a,q) &= q^{-\phi(a)}z^{-\widehat{I}(a)}\theta(z,q) \\ \theta(z^w,q) &= \theta(z,q) \end{aligned}$$

for $z\in\check{T}\otimes\mathbb{C}^\times$, $a\in\check{T}$, and $w\in W$.

These are precisely the equations which must be satsified by the character θ of a representation of the loop group LG of level c. In fact the Kac Character Formula identifies the $\mathbb{Z}((q))$-module of such theta functions (with integer cofficients) with the $\mathbb{Z}((q))$-span of characters of representations of LG of level c.

For example, consider the case that $G = Spin(2d)$. If we identify

$$\check{T}_{SO(2d)} \cong \mathbb{Z}^d$$

in the usual way, then

$$\check{T} \cong \{(m_1, \ldots, m_d) | \sum m_i \equiv 0 \mod 2\}.$$

The class c_2 corresponds to

$$\phi(m) = \frac{1}{2} \sum m_i^2$$

$$I(m, m') = \sum m_i m_i'.$$

The "basic" representation of $LSpin(2d)$ is a representation of level c_2, with character

$$\sigma(u_1, \ldots, u_d) = \prod_j \sigma(u_j, q).$$

Thus the product of sigma functions gives a section of $\mathcal{A}(c_2)$. Of course it is not a trivialization, but if $I(\sigma)$ is the ideal sheaf defined by the zeroes of χ, then σ does give a trivialization of $\mathcal{A}(c_2) \otimes I(\sigma)$.

In the course of proving the Theorem, I make a series of conjectures which amount to the following (see math.AT/0201092).

Let $B_{S^1}G$ be the classifying space for S^1-equivariant principal G-bundles. First of all, there should be a canonical isomorphism

$$E_{S^1}(B_{S^1}G) \cong (\check{T} \otimes C)/W.$$

Thus if V is a S^1-equivariant $G = Spin(2d)$-vector bundle over X, the map

$$X \to B_{S^1}G$$

classifying V gives rise to a map

$$E_{S^1}(X) \to (\check{T} \otimes C)/W.$$

By pulling back the line bundles $\mathcal{A}(c_2)$ and $I(\sigma)$ and the section σ, we obtain line bundles $\mathcal{A}(V)$ and $I(V)$ over $E_{S^1}(X)$, and a trivialization $\sigma(V)$ of $\mathcal{A}(V) \otimes I(V)$.

I further conjecture that there is a canonical isomorphism

$$I(V) \cong E_T(X^V)$$

of $E_T(X)$-modules, and that if V' is another equivariant spin bundle such that

$$c_2(V - V') = 0,$$

then a $BO\langle 8 \rangle_{S^1}$-structure on $V - V'$ determines a trivialization of $\mathcal{A}(V) \otimes \mathcal{A}(V)^{-1}$. In the presence of such a trivialization, then, $\sigma(V)/\sigma(V')$ is a trivialization of

$$\frac{\mathcal{A}(V) \otimes I(V)}{\mathcal{A}(V') \otimes I(V')} \cong \frac{I(V)}{I(V')} \cong \frac{E_{S^1}(X^V)}{E_{S^1}(X^{V'})}.$$

This is the sigma orientation; the theorem was proved by writing down the formulae implied by these conjectures.