ATTACHING CELLS

Let $X$ be a space, and let $A \subset X$ be a subspace. The pair $(X, A)$ is an \textit{NDR pair} if there are maps

$$
\begin{align*}
u &: X \to I \\
d &: X \times I \to X
\end{align*}
$$

such that

$$
\begin{align*}
A &= u^{-1}(0) \\
d(x, 0) &= x & x \in X \\
d(a, t) &= a & a \in A \\
d(x, 1) \in A &= u(x) < 1.
\end{align*}
$$

Let

$$
B = u^{-1}([0, 1]).
$$

The map $d$ induces a deformation retraction of the inclusion of $A$ in $B$: $A \hookrightarrow B$ is a homotopy equivalence.

\textbf{Lemma 1.} In this situation, the inclusion

$$(X, A) \hookrightarrow (X, B)$$

induces isomorphisms.

$$
H_q(X, A) \cong H_q(X, B).
$$

\textit{Proof.} The long exact sequences of the pairs give a ladder

$$
\begin{array}{ccc}
\hat{H}_q A & \longrightarrow & \hat{H}_q X \\
\cong & & \cong \\
\hat{H}_q B & \longrightarrow & \hat{H}_q X \\
& & \Downarrow \cong \\
& & \hat{H}_q (X, A)
\end{array}
$$

in which the indicated arrows are isomorphisms. The “5-lemma” gives the result. \hfill \Box

Now suppose that $f: A \to Y$ is a map. Let $Z = X \bigsqcup_f Y$ be the pushout in the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Z.
\end{array}
$$

\textbf{Lemma 2.} If $(X, A)$ is an NDR pair, then the map

$$
g|_{X-A} : X - A \to Z - Y
$$

is a homeomorphism.

\textit{Date: 22 January.}
Lemma 3. If \((X, A)\) is an NDR pair, then so is \((Z, Y)\).

**Sketch of proof:** We need to find maps \(\tilde{u} : Z \to I\) and \(\tilde{d} : Z \times I \to Z\) which satisfy the conditions of an NDR pair.

First \(\tilde{u}\). Consider the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & Z \\
\end{array}
\]

Since the square is a pushout, the dotted arrow exists. For \(\tilde{d}\), consider the diagram

\[
\begin{array}{ccc}
A \times I & \xrightarrow{f \times I} & Y \times I \\
\downarrow & & \downarrow \\
X \times I & \xrightarrow{g \times I} & Z \times I \\
\end{array}
\]

Again this diagram is a pushout (check!), and the dotted arrow exists. It is easily seen to have the right properties.

**Proposition 4.** In the situation above, the map

\[
H_q g : H_q(X, A) \to H_q(Z, Y)
\]

is an isomorphism for all \(q\).

**Proof.** Let \(B = u^{-1}[0, 1]\) be the deformation neighborhood of \(A\). Note that

\[
\tilde{u}^{-1}[0, 1] = Y \cup g(B)
\]

is the deformation neighborhood of \(Y\). Lemma ?? gives isomorphisms

\[
H_q(X, A) \xrightarrow{i} H_q(X, B) H_q(Z, Y) \xrightarrow{j} H_q(Z, Y \cup g(B)).
\]
The virtue of the neighborhood $B$ over $A$ is that we can excise $A$ from it: we have a commutative diagram

$$
\begin{array}{c}
H_q(X - A, B - A) \xrightarrow{\cong} H_q(X, B) \\
\downarrow H_q \cong \downarrow \alpha \\
H_q(Z - Y, g(B - A)) \xrightarrow{\cong} H_q(Z, Y \cup g(B)).
\end{array}
$$

Lemma ?? gives the left vertical isomorphism. It follows that $\alpha$ is an isomorphism.

Now consider the commutative diagram

$$
\begin{array}{c}
H_q(X, A) \xrightarrow{i} H_q(X, B) \\
\downarrow \cong \downarrow \alpha \\
H_q(Z, Y) \xrightarrow{j} H_q(Z, Z \coprod_y B).
\end{array}
$$

Three of the arrows are isomorphisms, so the fourth is, too.

Applying the proposition in the case that $Y = \ast$, we get

**Corollary 5.** If $(X, A)$ is an NDR pair, then the natural map

$$H_q(X, A) \to H_q(X/A, \ast) = \tilde{H}_q(X/A)$$

is an isomorphism.

This is charming, but if you apply the Barratt-Whitehead Lemma to the diagram

$$
\begin{array}{c}
\tilde{H}_q A \longrightarrow \tilde{H}_q X \longrightarrow H_q(X, A) \longrightarrow \tilde{H}_{q-1} A \\
\downarrow \downarrow \downarrow \cong \downarrow \\
\tilde{H}_q Y \longrightarrow \tilde{H}_q Z \longrightarrow H_q(Z, Y) \longrightarrow \tilde{H}_{q-1} Y
\end{array}
$$

you get a long exact sequence

$$\ldots \tilde{H}_q A \to \tilde{H}_q Y \oplus \tilde{H}_q X \to \tilde{H}_q Z \to \ldots.$$

Let $D^n$ be the closed disk of vectors in $\mathbb{R}^n$ of length less than or equal to 1.

**Lemma 6.** $(D^n, S^{n-1})$ is an NDR pair.

**Proof.** Let

$$u(x) = \begin{cases} 
2(1 - \|x\|) & \|x\| \geq 1/2 \\
0 & \|x\| < 1/2.
\end{cases}$$

Let

$$s : \mathbb{R} \to \mathbb{R}$$

be a given by the formula

$$s(r) = \begin{cases} 
0 & r \leq \frac{1}{4} \\
4(r - \frac{1}{4}) & \frac{1}{4} < r \leq \frac{1}{2}, \\
1 & \frac{1}{2} < r.
\end{cases}$$
Let
\[ d(x, t) = s(\|x\|)t \frac{x}{\|x\|} + (1 - s(\|x\|))tx \]
for \( x \neq 0 \), and \( d(0, t) = 0 \). It is easy to see that \( d \) is continuous, and that \( u \) and \( d \) have the desired properties.

Applying our long exact sequence in the case \((X, A) = (D^n, S^{n-1})\), we find isomorphisms
\[
\tilde{H}_q Z \cong \tilde{H}_q Y \quad q \geq n + 1
\]
\[
\tilde{H}_q Z \cong \tilde{H}_q Y \quad q \leq n - 2
\]
\[
\tilde{H}_{n-1} Z \cong \text{Coker } \tilde{H}_{n-1} f
\]
and a short exact sequence
\[
0 \rightarrow \tilde{H}_n Y \rightarrow \tilde{H}_n Z \rightarrow \text{Ker } \tilde{H}_{n-1} f \rightarrow 0.
\]
In the applications, one often has \( \tilde{H}_n Y = 0 \) and so
\[
\tilde{H}_n Z \cong \text{Ker } \tilde{H}_{n-1} f.
\]

**Example 1.** The real projective space \( \mathbb{R}P^2 \) fits into a pushout diagram
\[
\begin{array}{ccc}
S^1 & \rightarrow & S^1 \\
\downarrow & & \downarrow \\
D^2 & \rightarrow & \mathbb{R}P^2.
\end{array}
\]
It follows that
\[
\tilde{H}_q \mathbb{R}P^2 = 0 \quad q \neq 1
\]
\[
\tilde{H}_1 \mathbb{R}P^2 \cong \mathbb{Z}/2.
\]

**Example 2.** Let \( Y = S^1 \vee S^1 \). Let \( a, b : S^1 \rightarrow Y \) be the inclusion of the two circles in the bouquet. The torus \( T^2 \) fits into a pushout diagram
\[
\begin{array}{ccc}
S^1 & \rightarrow & Y \\
\downarrow & & \downarrow \\
D^2 & \rightarrow & T^2.
\end{array}
\]
Now the map \( aba^{-1}b^{-1} \) is zero in \( H_1 \), because \( H_1 \) is abelian. It follows that
\[
\tilde{H}_q T^2 = 0 \quad q \neq 1, 2
\]
\[
\tilde{H}_1 T^2 \cong \mathbb{Z}^2
\]
\[
\tilde{H}_2 T^2 \cong \mathbb{Z}.
\]

**Exercise 1.** Calculate \( \tilde{H}_q T^n \) and \( \tilde{H}_q \mathbb{R}P^n \) for \( n > 2 \).

Hint: We calculated the homology of \( P^2 \) by attaching a 2-cell to the circle by the map \( z^2 \). The map \( z^2 \) in this context may be understood as the projection of \( S^1 \) unto \( S^1 / x \sim -x \cong \mathbb{R}P^1 \). More generally, there will always be a projection map
\[
S^n \rightarrow \mathbb{R}P^n.